

ON THE SYMPLECTIC COHOMOLOGY OF LOG CALABI–YAU SURFACES

JAMES PASCALEFF

ABSTRACT. This article studies the symplectic cohomology of affine algebraic surfaces that admit a compactification by a normal crossings anticanonical divisor. Using a toroidal structure near the compactification divisor, we describe the complex computing symplectic cohomology, and compute enough differentials to identify a basis for the degree-zero part of the symplectic cohomology. This basis is indexed by integral points in a certain integral affine manifold, providing a relationship to the theta functions of Gross–Hacking–Keel. Also included is a discussion of wrapped Floer cohomology of Lagrangian submanifolds and a description of the product structure in a special case.

CONTENTS

1. Introduction	2
1.1. Acknowledgments	5
2. Symplectic cohomology and holomorphic curves	5
2.1. Conventions	5
2.2. Holomorphic curves	7
2.3. Floer cohomology	8
2.4. Example of the complex torus	10
3. Log Calabi–Yau surfaces	11
3.1. Basics	11
3.2. Examples	13
4. Construction of the Liouville domain	15
4.1. Basic Liouville structure on an affine variety	15
4.2. Refinements of the basic construction	16
4.3. Reeb dynamics and Liouville classes	17
5. The affine manifold	23
5.1. Initial data	24
5.2. The local model for nodes	24
5.3. The local model for divisors	24
5.4. Affine charts and gluing	26
6. The differential on symplectic cohomology	26
6.1. Generators of symplectic cohomology	27
6.2. Holomorphic cylinders	29
6.3. Neck stretching	31
6.4. Computation of differentials	32
6.5. Continuation maps	34
7. Wrapped Floer cohomology	35
7.1. Lagrangian sections	36
7.2. The case of the affine plane minus a conic	37

Date: 18 April 2013.

2010 Mathematics Subject Classification. Primary 53D40; Secondary 53D37, 14J33.

The author was partially supported by NSF grant DMS-0636557.

1. INTRODUCTION

Log Calabi-Yau manifolds have a rich symplectic geometry. They are the subject of a mirror symmetry conjecture: in favorable situations, there is a mirror variety U^\vee such that the Floer theory of $U = Y \setminus D$ is reflected in the algebraic geometry of U^\vee . For instance, as conjectured by Gross–Hacking–Keel [16], one expects to find the ring of regular functions on U^\vee sitting inside the symplectic cohomology of U . Furthermore, the symplectic cohomology should come with a natural basis corresponding to a collection of “theta functions” on U^\vee [16, 17]. This article contains results towards this conjecture in complex dimension two.

Symplectic cohomology is a version of Hamiltonian Floer homology that applies to open symplectic manifolds. In particular, given a smooth complex affine variety U , we can equip U with an exact symplectic structure coming from a projective compactification Y by an ample divisor D . The symplectic cohomology does not depend on the choices involved [22, §4b], so we may speak of the symplectic cohomology $SH^*(U)$ of the affine variety U .

Let us briefly describe the geometry that is involved in the definition of $SH^*(U)$. It is the cohomology of a cochain complex $SC^*(U)$, which for an appropriate choice of Hamiltonian function, contains both the ordinary cohomology of U as well as generators corresponding to periodic Reeb orbits in a contact hypersurface at infinity. Both the cochain complex $SC^*(U)$ and its cohomology $SH^*(U)$ are in general infinite dimensional, even in a single degree. The differential on the complex counts maps of cylinders into U satisfying an inhomogeneous pseudo-holomorphic map equation.

Symplectic cohomology is interesting from several points of view in symplectic topology. One application is the Weinstein conjecture: if a certain canonical map $H^*(U) \rightarrow SH^*(U)$ is not an isomorphism, then the contact hypersurface at infinity must contain a periodic Reeb orbit. However, the motivating interest in this paper comes from homological mirror symmetry. Symplectic cohomology is the closed-string sector of a two-dimensional field theory that forms the A-side of one version of HMS for an open symplectic manifold U , whose open-string sector is the wrapped Fukaya category of Abouzaid–Seidel [5]. The B-side is an appropriate category of sheaves \mathcal{C} on a dual space U^\vee , for which the closed-string sector is the Hochschild cohomology $HH^*(\mathcal{C})$.

This line of thinking leads to interesting predictions about the symplectic cohomology of U in terms of the algebraic geometry of U^\vee . We beg the reader to bear with this rather conjectural paragraph. Let us adopt the ansatz that the mirror U^\vee is a smooth affine algebraic variety over a base field \mathbb{K} . Homological mirror symmetry implies that the derived wrapped Fukaya category $D^b\mathcal{W}(U)$ is equivalent to the derived category of coherent sheaves $D^b\mathrm{Coh} U^\vee$, and hence that their Hochschild cohomologies $SH^*(U)$ and $HH^*(U^\vee)$ are isomorphic. But now

$$(1) \quad HH^p(U^\vee) \cong \bigoplus_{i=0}^p H^i(U^\vee, \Lambda^{p-i} T_{U^\vee}) \cong H^0(U^\vee, \Lambda^p T_{U^\vee}),$$

where the first isomorphism is Hochschild–Kostant–Rosenberg, and the second holds because of the ansatz that U^\vee is affine (Cartan–Serre Theorem B). Setting $p = 0$,

$$(2) \quad SH^0(U) \cong HH^0(U^\vee) \cong H^0(U^\vee, \mathcal{O}_{U^\vee}).$$

Since U^\vee is affine, $U^\vee = \mathrm{Spec} H^0(U^\vee, \mathcal{O}_{U^\vee})$. Thus we find

$$(3) \quad U^\vee \cong \mathrm{Spec} SH^0(U),$$

giving us a way to reconstruct the mirror U^\vee from the symplectic cohomology of U .

This argument raises some basic questions. First of all, it requires that $SH^*(U)$ is a ring. This ring structure is well-known in Hamiltonian Floer homology as the pair-of-pants product. More surprisingly, it implies that $SH^*(U)$ is \mathbb{Z} -graded. The \mathbb{Z} -grading may not seem like a central issue from a symplectic topology point of view, but it is closely related to the Calabi–Yau nature of mirror symmetry. The symplectic geometry definition of $SH^*(U)$ yields only $\mathbb{Z}/2\mathbb{Z}$ -grading, and in general this can be lifted to a \mathbb{Z} -grading when $c_1(U) = 0$. Even then, the actual \mathbb{Z} -grading depends on a trivialization of the canonical bundle of U . Therefore, in order to make the isomorphism (2) work, the class of symplectic manifolds U under consideration must be restricted and extra data may need to be included on the A-side.

The relevant definition here is that of a log Calabi–Yau manifold. In this definition, note that a normal crossings divisor is necessarily effective and reduced.

Definition 1. Let U be a smooth complex quasiprojective surface. U is *log Calabi–Yau* if there is a smooth projective surface Y containing an at worst nodal anticanonical divisor D such that $U \cong Y \setminus D$. If D is actually nodal, (Y, D) is known as a *Looijenga pair*, or is said to have *maximal boundary*.

It turns out that the divisor D in this situation must either be a smooth genus one curve, a nodal genus one curve, or a cycle of rational curves meeting at nodes (Lemma 3.1).

Since D is anticanonical, effective, and reduced, there is a meromorphic volume form Ω with simple poles along D (and no other zeros or poles). Thus the restriction $\Omega|_U$ is a nowhere-vanishing complex volume form on U that gives a specific trivialization of the canonical bundle, showing $c_1(U) = 0$. It turns out that the homotopy class of this trivialization does not depend on the choice of compactification (Y, D) (Lemma 3.2).

To illustrate the sort of restriction this places on U , consider complex dimension one. The condition that D must be effective and anticanonical implies $Y \cong \mathbb{P}^1$, and that the support of D is at most two points. Thus U is either \mathbb{C} or \mathbb{C}^\times . Both of these satisfy $c_1(U) = 0$, as does any punctured curve, but only $U = \mathbb{C}^\times$ (with $D = 0 + \infty$) is log Calabi–Yau in our definition, since the divisor $D = 2\infty$ is not reduced. In terms of symplectic cohomology, the contrast is stark, as $SH^*(\mathbb{C}) = 0$ (which is the cohomology of the empty set), while $SH^*(\mathbb{C}^\times) \cong \mathbb{K}[x, x^{-1}, \theta]$, where x is in degree zero and θ is in degree one. Observe that $SH^0(\mathbb{C}^\times)$ is isomorphic to $\mathbb{K}[x, x^{-1}]$, the coordinate ring of the expected mirror \mathbb{G}_m .

In this paper, we make two other assumptions, namely that U is affine, and so is an exact symplectic manifold with a well-defined symplectic cohomology, and also that the complex dimension is two, which is the first interesting low-dimensional case. This case was also studied extensively in an algebro-geometric context by Gross–Hacking–Keel [16], who proposed that the degree zero symplectic cohomology should be the coordinate ring of the mirror, in favorable cases. They approach the construction of the mirror from another direction, and actually define the coordinate ring of the mirror using tropical geometry (an essentially combinatorial theory). Our goal is to verify some of their predictions in terms of symplectic geometry and Floer theory, so we actually work with maps of Riemann surfaces into the symplectic manifold.

The first step towards verifying the Gross–Hacking–Keel prediction is to compute the symplectic cohomology additively. Looking at what they have, we find that $\Gamma(U^\vee, \mathcal{O}_{U^\vee})$ is additively generated by a basis $\{\theta_p \mid p \in U^{\text{trop}}(\mathbb{Z})\}$, where the elements are indexed by integral points of a certain integral affine manifold U^{trop} associated to U (we also describe a construction of this manifold). These elements θ_p are the so-called theta functions on the mirror U^\vee [16, 17].

By actually looking at the periodic Hamiltonian orbits used to define symplectic cohomology, we also find an additive basis for symplectic cohomology whose indexing set is in correspondence with $U^{\text{trop}}(\mathbb{Z})$, via a geometrically natural bijection.

Theorem 1.1. *Let U be an affine log Calabi–Yau surface with maximal boundary. We construct a set of cochains $\{\theta_p \mid p \in U^{\text{trop}}(\mathbb{Z})\}$. They have degree zero in the \mathbb{Z} -grading on symplectic cohomology induced by a log Calabi–Yau compactification (Y, D) , and span the degree zero cochains. They are closed and represent linearly independent classes in cohomology. Thus*

$$(4) \quad SH^0(U) = \text{span}\{\theta_p \mid p \in U^{\text{trop}}(\mathbb{Z})\}$$

The element θ_p is defined with reference a particular Liouville structure on U , which is deformation equivalent to the Stein structure, and a particular Hamiltonian $H : U \rightarrow \mathbb{R}$. With this in mind, θ_p is the periodic orbit of degree zero arising from perturbation of a periodic torus of the Hamiltonian flow. Such tori are in a geometrically natural bijection with $U^{\text{trop}}(\mathbb{Z})$.

Let us outline briefly how this computation goes. In section 4, we construct a Liouville structure on U , which admits a Lagrangian torus fibration over the cylindrical end of U . This construction is based largely on [22, §4], with some tweaks using the extra symmetry of our situation. This structure has another convenient property, namely the contact manifold on which the cylindrical end is modeled satisfies a certain “convexity” condition.¹ With an appropriate Hamiltonian, the periodic orbits can be explicitly described. In fact, there are entire tori that are periodic for the Hamiltonian flow, so this is a Morse–Bott situation. After small time-dependent perturbation of the Hamiltonian, these tori break up into several non-degenerate orbits of various degrees. The orbit of degree zero is the element θ_p .

The crucial point is to understand the differential, and in particular to show that θ_p is closed, so that it is actually a cocycle rather than just a cochain element. For the definitions see section 2. The convexity condition is important for understanding the holomorphic curves that contribute to the differential, as it actually allows us to show that certain moduli spaces are empty for energy reasons. At this point in the argument we use some ideas coming from symplectic field theory, adapted to the framework of Hamiltonian Floer homology, namely a neck-stretching argument due to Bourgeois and Oancea [8], and an adaptation of technique developed by Bourgeois and Colin [7] to compute contact homology of toroidal manifolds. To get from these techniques to algebraic fact that θ_p is closed, we use the way the differential interacts with the Batalin–Vilkovisky operator and the pair-of-pants product, so we also obtain some limited information about these operations as well. This is done in section 6.

Above we described how homological mirror symmetry applied to Hochschild cohomology leads to the expectation that $SH^0(U)$ is isomorphic the ring of global functions on the mirror U^\vee . However, there is another, more obvious way to extract this ring from the category of coherent sheaves on U^\vee , namely as $\text{Hom}(\mathcal{O}_{U^\vee}, \mathcal{O}_{U^\vee})$. Under the ansatz that U^\vee is affine, there are no higher Ext-groups, so we should expect to be able to find a Lagrangian submanifold L in U corresponding to \mathcal{O}_{U^\vee} , having wrapped Floer cohomology $HW^*(L, L)$ concentrated in degree zero, and such that $SH^0(U)$ is isomorphic to $HW^0(L, L)$. In section 7 we consider Lagrangian submanifolds with this properties, which are candidates for the mirror to \mathcal{O}_{U^\vee} . In the case where U is the complement of a smooth conic in \mathbb{C}^2 , we can combine the results of this paper with those of [19] to understand the ring structure and show that

$$(5) \quad SH^0(U) \cong \mathbb{K}[x, y][(xy - 1)^{-1}]$$

Section 3 contains some basic results on log Calabi-Yau surfaces. Section 5 describes the affine manifold U^{trop} from a topological viewpoint.

¹This condition is independent of the contact condition, which may also be regarded as a convexity condition.

The paper is organized so that sections 2, 4, and 6 are a continuous thread of argument. Sections 3 and 5 can be read as interludes describing the algebro-geometric and piecewise-linear context of our study.

1.1. Acknowledgments. The author wishes to thank Sean Keel for many helpful discussions of his joint work with Mark Gross and Paul Hacking, which formed the background of this work. He also thanks Mohammed Abouzaid, Matthew Strom Borman, Luis Diogo, Sheel Ganatra, Mark McLean, Tim Perutz, and Paul Seidel for many helpful discussions and suggestions on techniques to compute symplectic cohomology. In particular, the use of the Batalin–Vilkovisky operator that appears in Proposition 6.13 was suggested by Paul Seidel. This work was done while the author held an RTG postdoctoral fellowship (NSF grant DMS-0636557).

2. SYMPLECTIC COHOMOLOGY AND HOLOMORPHIC CURVES

This section reviews material about pseudo-holomorphic curves in the case of manifolds with contact-type boundary and cylindrical ends, and a description of the versions of Floer cohomology that we use. We hope that this section will make the paper more self-contained for algebro-geometrically minded readers.

2.1. Conventions. In this section, we set out the conventions for symplectic manifolds and symplectic cohomology. In large part our conventions follow [22, 20].

Definition 2. An *exact symplectic form* on a manifold M is an exact non-degenerate two-form ω . Thus

$$(6) \quad \omega = d\lambda$$

for some one-form λ , which is called a *Liouville one-form*. The corresponding *Liouville vector field* Z is defined by duality with respect to ω :

$$(7) \quad \iota_Z \omega = \omega(Z, \cdot) = \lambda$$

Thus the one-form λ determines the two-form ω and the vector field Z .

A *Liouville domain* [22] is a compact manifold with boundary, equipped with a one-form λ , such that the two-form ω is symplectic, and the vector field Z points strictly outward along the boundary.

Definition 3. Let $L \subset M$ be a Lagrangian submanifold. The Liouville one-form λ defines a class $[\lambda|_L] \in H^1(L, \mathbb{R})$, called the *Liouville class*.

Definition 4. Let (M, ω) be a symplectic manifold. An almost complex structure J on M is *compatible* with ω if the bilinear form g defined by

$$(8) \quad g(X, Y) = \omega(X, JY)$$

is symmetric and positive definite at every point of M . Thus g is a Riemannian metric associated to the choice of J .

Definition 5. If $H : M \rightarrow \mathbb{R}$ is a differentiable function, the *hamiltonian vector field* X_H associated to H is defined by the relation

$$(9) \quad -\iota_{X_H} \omega = \omega(\cdot, X_H) = dH$$

In the presence of a compatible almost complex structure J and associated metric g , we may take the gradient ∇H with respect to g . As a consequence of our conventions, this is connected to X_H by

$$(10) \quad X_H = J\nabla H$$

Example 1. Let $M = \mathbb{C}$ be the complex affine line, coordinatized by $z = x + iy$. We take the Euclidean structures $\omega = \frac{i}{2}dz \wedge d\bar{z} = dx \wedge dy$, J = multiplication by i , $g = dx^2 + dy^2$. For λ we choose

$$(11) \quad \lambda = d^c \left(\frac{1}{4}|z|^2 \right) = \frac{i}{4}(z d\bar{z} - \bar{z} dz) = \frac{1}{2}(x dy - y dx)$$

where we have taken advantage of the complex analytic structure of M to write λ in terms of a Kähler potential. Here $d^c = -i(\partial - \bar{\partial})$ as in, for example, [26].

The corresponding Liouville vector field is

$$(12) \quad Z = \frac{1}{2} \left(z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right) = \frac{1}{2} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)$$

Since the vector field Z points radially outward, we find that any disk $\{|z| \leq R\}$ becomes a Liouville domain when equipped with the restrictions of these structures.

Now consider the function $H = \frac{1}{2}(x^2 + y^2)$. We have

$$(13) \quad \nabla H = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

$$(14) \quad X_H = J\nabla H = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

Thus the flow of X_H rotates the plane about the origin in the counterclockwise direction with period 2π .

Lemma 2.1. *If (M, λ) is a Liouville domain of real dimension $2n$, then $\alpha \in \Omega^1(\partial M)$ defined by*

$$(15) \quad \alpha = \lambda|_{\partial M}$$

is a contact one-form on the boundary ∂M , which is to say $\alpha \wedge (d\alpha)^{2n-2}$ is a volume form on ∂M .

Definition 6. Let (N, α) be a contact manifold with contact one-form α . We have the contact distribution $\xi = \ker \alpha$. The *Reeb vector field* is defined by the conditions $\iota_R d\alpha = 0$, $\alpha(R) = 1$. The *symplectization* of (N, α) , is an exact symplectic structure on $N \times \mathbb{R}$. Letting ρ denote the coordinate on the \mathbb{R} factor, the Liouville one-form, symplectic form, and Liouville vector field are

$$(16) \quad \lambda = e^\rho \alpha$$

$$(17) \quad \omega = e^\rho(d\rho \wedge \alpha + d\alpha)$$

$$(18) \quad Z = \frac{\partial}{\partial \rho}$$

Symplectic manifolds of the form $N \times \mathbb{R}$ are also called *cylindrical symplectic manifolds*.

An almost complex structure J on $N \times \mathbb{R}$ is *cylindrical* if it respects the product structure in the following way: with respect to the decomposition

$$(19) \quad T(N \times \mathbb{R}) = TN \oplus \langle Z \rangle = \xi \oplus \langle R \rangle \oplus \langle Z \rangle$$

we require that J preserves ξ and $J(Z) = R$. Thus $J|_\xi$ is an almost complex structure on ξ . The full structure J is compatible with ω if and only if $J|_\xi$ is compatible with $d\alpha|_\xi$, which is a symplectic form on ξ .

Definition 7. A $(n-1)$ -dimensional submanifold $\Lambda \subset N$ is called *Legendrian* if $T\Lambda \subset \xi = \ker \alpha$. A n -dimensional submanifold $L \subset N$ is called *pre-Lagrangian* if $d\alpha|_L = 0$. The one-form α then defines a class $[\alpha|_L] \in H^1(L, \mathbb{R})$ also called the *Liouville class*. Observe that in the symplectization L lifts to $L \times \{0\} \subset N \times \mathbb{R}$ which is a Lagrangian submanifold with the same Liouville class.

Lemma 2.2. *Let $N \times \mathbb{R}$ be a cylindrical symplectic manifold, and let $H : N \times \mathbb{R} \rightarrow \mathbb{R}$ be a function which depends on ρ only. Thus $H(x, \rho) = h(e^\rho)$ for some function $h : (0, \infty) \rightarrow \mathbb{R}$. Then the Hamiltonian vector field X_H is tangent to each slice $N \times \{\rho\}$, and is proportional to the Reeb vector field R :*

$$(20) \quad X_H = h'(e^\rho)R$$

If J is a compatible cylindrical almost complex structure, with corresponding metric g , then $X_H = J\nabla H$, where

$$(21) \quad \nabla H = h'(e^\rho) \frac{\partial}{\partial \rho}$$

Definition 8. Let (M, λ) be an exact symplectic manifold, and $H : M \rightarrow \mathbb{R}$ a hamiltonian function. The symplectic action of a loop $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow M$, whose domain is parametrized by $t \in [0, 1]$, is given by

$$(22) \quad \mathcal{A}(\gamma) = - \int_{\mathbb{R}/\mathbb{Z}} \gamma^* \lambda + \int_0^1 H(\gamma(t)) dt$$

The critical points of this action functional are those loops γ such that $-\iota_{\dot{\gamma}} d\lambda = dH$, which in light of definition 5 means $\dot{\gamma} = X_H$. In other words, the critical points are 1-periodic orbits of X_H .

2.2. Holomorphic curves. In this section we recall some elementary facts about inhomogeneous pseudo-holomorphic maps that will be used in the paper. Throughout, let C be a Riemann surface with complex structure j .

Definition 9. Let (M, J) be an almost complex manifold. A map $u : C \rightarrow M$ is *pseudo-holomorphic* if $J \circ du = du \circ j$.

Definition 10. Let (M, ω, J) be a symplectic manifold with compatible almost complex structure. Let $H : M \rightarrow \mathbb{R}$ be a hamiltonian function, with hamiltonian vector field $X_H = J\nabla H$. Let $\beta \in \Omega^1(C)$ be a one-form. A map $u : C \rightarrow M$ is an *inhomogeneous pseudo-holomorphic map* if

$$(23) \quad J \circ (du - X_H \otimes \beta) = (du - X_H \otimes \beta) \circ j$$

Let $M = N \times \mathbb{R}$ is a cylindrical symplectic manifold and J is a cylindrical almost complex structure. Then we may write any map $u : C \rightarrow N \times \mathbb{R}$ as $u = (f, a)$, where $f : C \rightarrow N$ and $a : C \rightarrow \mathbb{R}$. Let $\pi_\xi : TN \rightarrow \xi$ denote the projection whose kernel is the Reeb field R , and let $\pi_R : TN \rightarrow \langle R \rangle$ denote the complementary projection. Observe that $\pi_R(X) = \alpha(X)R$. Let J_ξ denote the ξ component of J .

Proposition 2.3. *The map $u = (f, a) : C \rightarrow N \times \mathbb{R}$ is pseudo holomorphic iff*

$$(24) \quad J_\xi \circ \pi_\xi \circ df = \pi_\xi \circ df \circ j$$

$$(25) \quad J \circ da = \pi_R \circ df \circ j$$

If we identify da with a one-form on C , the second equation may be expressed as

$$(26) \quad da = \alpha \circ df \circ j = (f^* \alpha) \circ j$$

The system says that $\pi_\xi \circ df : TC \rightarrow \xi$ is complex linear, and the one-form $(f^ \alpha) \circ j$ is exact, with a being an antiderivative.*

Proof. Clear by decomposing the tangent space to the target as in (19). \square

Proposition 2.4. *Let $H = h(e^\rho)$ be a hamiltonian function on $N \times \mathbb{R}$ that depends only on the \mathbb{R} -coordinate ρ , and let $\beta \in \Omega^1(C)$. Then $u = (f, a) : C \rightarrow N \times \mathbb{R}$ be a solution of (23) iff*

$$(27) \quad J_\xi \circ \pi_\xi \circ df = \pi_\xi \circ df \circ j$$

$$(28) \quad J \circ da = (\pi_R \circ df - X_H \otimes \beta) \circ j$$

If we identify da with a one-form on C , the second equation may be expressed as

$$(29) \quad da = (\alpha \circ df - \alpha(X_H)\beta) \circ j$$

Note that the expression $\alpha(X_H) = h'(e^\rho)$ depends functionally on a but not on f .

Proof. First observe that the condition on H implies that X_H is proportional to R , thus $\pi_\xi(X_H) = 0$, and $\pi_R(X_H) = X_H = \alpha(X_H)R$. By considering the ξ -component, we obtain the first equation. The R component of $du - X_H \otimes \beta$ is $\pi_R \circ df - X_H \otimes \beta$, while the Z component is da . \square

By comparing these two propositions we see that in both cases, we see that in both cases for a map $u = (f, a)$ to solve the equation it is necessary that $\pi_\xi \circ df$ is complex linear, in which case a can be more or less reconstructed from f if it exists. This motivates the definition of a pseudo holomorphic curve in a contact manifold.

Definition 11. Let (N, α) be a contact manifold, and J_ξ an almost complex structure on $\xi = \ker \alpha$ compatible with $d\alpha$. A map $f : C \rightarrow N$ is called *pseudo holomorphic* if

$$(30) \quad J_\xi \circ \pi_\xi \circ df = \pi_\xi \circ df \circ j$$

The next proposition expresses the familiar principle that “holomorphic curves are symplectic.”

Proposition 2.5. *Let $f : C \rightarrow N$ be a pseudo holomorphic map. Then $f^*d\alpha$ is a non-negative 2-form (with respect to the complex orientation of C). Furthermore $f^*d\alpha$ can only vanish at a point where $\pi_\xi \circ df : TC \rightarrow \xi$ vanishes as a linear transformation, or equivalently df maps TC into the line spanned by R .*

Proof. For any vector $v \in TC$, we have an oriented basis $\langle v, jv \rangle$ of TC . We compute

$$(31) \quad \begin{aligned} f^*d\alpha(v, jv) &= d\alpha(df(v), df \circ j(v)) = d\alpha(\pi_\xi \circ df(v), \pi_\xi \circ df \circ j(v)) \\ &= d\alpha(\pi_\xi \circ df(v), J_\xi \circ \pi_\xi \circ df(v)) = \|\pi_\xi \circ df(v)\|_{g_\xi}^2 \geq 0 \end{aligned}$$

and the expression can only vanish if $\pi_\xi \circ df(v) = 0$, which is to say that $df(v)$ is proportional to R . \square

2.3. Floer cohomology. The moduli spaces of inhomogeneous pseudo-holomorphic maps to a fixed symplectic target M may be used to setup a TQFT-type structure of which the symplectic cohomology is a part. Fix a base field \mathbb{K} . Assume that M comes with a cylindrical end with natural coordinate ρ . The relevant references are [22, 23, 20].

Definition 12. A Hamiltonian $H^m : M \rightarrow \mathbb{R}$ that is of the form $H^m = me^\rho + C$ for large ρ is said to have *linear of slope m at infinity*. A Hamiltonian H^Q that is of the form $H^Q = C(e^\rho)^2 + D$ for large ρ is said to be *quadratic at infinity*.

The rough idea is that we can work either with a quadratic Hamiltonian H^Q , or with a family of linear Hamiltonians $\{H^m\}$, and take the limit as m goes to infinity to eliminate the dependence on m . To explain the latter version, let $m \in \mathbb{R}$ be a number so that all 1-periodic orbits of X_{H^m} lie in a compact subset of M . Equivalently, we require that m is not equal to the period of any Reeb orbit in the contact hypersurface $\{\rho = 0\}$. Let J be a compatible almost complex structure that is cylindrical for large ρ .

Take a time-dependent perturbation $K : S^1 \times M \rightarrow \mathbb{R}$ of H^m such that $K(t, x) = H^m(x)$ for x outside a compact subset of M , which is such that all the 1-periodic orbits of X_K are non-degenerate. These 1-periodic orbits form a basis of the cochain complex $CF^*(H^m)$. The differential d , of degree 1, is defined as follows. Take a time-dependent family of almost complex structures $J(t, x)$ that are equal to the given cylindrical $J(x)$ outside of a compact subset, which is chosen so as to make the moduli space of Floer trajectories regular. This is the moduli space of inhomogeneous pseudo-holomorphic maps $u(s, t) : \mathbb{R} \times S^1 \rightarrow M$ satisfying the equation

$$(32) \quad \begin{cases} \partial_s u + J(t, u)(\partial_t u - X_K(t, u)) = 0 \\ \lim_{s \rightarrow \pm\infty} u(s, t) = \gamma_{\pm}(t) \end{cases}$$

where γ_{\pm} are generators of $CF^*(H^m)$. The signed count of solutions to this equation (modulo the \mathbb{R} -translation action that shifts the coordinate s on the domain) yields the coefficient of γ_- in $d(\gamma_+)$. The cohomology of this cochain complex is the Floer cohomology $HF^*(H^m)$.

The Floer cohomologies $HF^*(H^m)$ for various values of the slope parameter m are not isomorphic, but are related by *continuation maps*, which count solutions to Floer's equation where the inhomogeneous term $X_K(t, u)$ now depends on s as well (breaking the \mathbb{R} -translation symmetry) and it interpolates between the corresponding terms used to define $CF^*(H^m)$ (at $s \gg 0$) and $CF^*(H^{m'})$ (at $s \ll 0$). Assuming that $m' \geq m$ and the interpolation satisfies a monotonicity condition, this leads to a chain map (of degree 0)

$$(33) \quad c_{m,m'} : CF^*(H^m) \rightarrow CF^*(H^{m'})$$

The continuation maps form a directed system, and by passing to the direct limit (category-theoretical colimit), we get a definition of the symplectic cohomology of M .

$$(34) \quad SH^*(M) = \lim_{m \rightarrow \infty} HF^*(H^m)$$

We will have use for a few other parts of the TQFT structure, namely the Batalin-Vilkovisky (BV) operator and the product. The BV operator δ also counts cylinders, but where the perturbation data are allowed to vary in a one parameter family, parametrized by $r \in S^1$. To define the BV operator $\delta : CF^*(H^m) \rightarrow CF^{*-1}(H^m)$, we start with the same time-dependent perturbations $K(t, x)$ and $J(t, x)$ used to define the differential. On the $s \ll 0$ end of the domain cylinder, we use these structures, while on the $s \gg 0$ end of the domain, we use $K(t+r, x)$ and $J(t+r, x)$. The asymptotic condition at the $s \gg 0$ end then becomes $\lim_{s \rightarrow \infty} u(s, t) = \gamma_+(t+r)$. Counting solutions that are isolated even as the parameter r is allowed to vary yields the degree -1 map δ . One finds that δ is a chain map, and that compatibility of the δ for various m yields a BV operator $\Delta : SH^*(M) \rightarrow SH^{*-1}(M)$. A useful property is that Δ vanishes on the image of the canonical map $H^*(M) \rightarrow SH^*(M)$.

The product is the TQFT operation associated to the pair of pants. It defines a map

$$(35) \quad HF^*(H^m) \otimes HF^*(H^{m'}) \rightarrow HF^*(H^{m''})$$

as long as $m'' \geq m+m'$. Passing to the limit as $m \rightarrow \infty$, this induces a product on $SH^*(M)$.

In the setup where a quadratic Hamiltonian is used, the periodic orbits will not in general be contained in any compact set, meaning that in order to achieve non-degeneracy and transversality, the perturbation of the Hamiltonian and complex structure cannot necessarily be compactly supported. This makes the compactness for pseudo-holomorphic curves more subtle. Ritter [20] provides two approaches for overcoming this difficulty and defining the TQFT structure using quadratic Hamiltonians.

2.4. Example of the complex torus. In the rest of the paper we are interested in complex dimension two, but in this section let the dimension be general n . Let $N \cong \mathbb{Z}^n$ be a lattice, $M = \text{Hom}(N, \mathbb{Z})$. Let $T = N \otimes_{\mathbb{Z}} \mathbb{C}^{\times} \cong (\mathbb{C}^{\times})^n$ be the complex torus. Let z_i be a set of coordinates on T in bijection with a basis of M . For the purpose of grading Floer cohomology, we use the complex volume form $\Omega = \prod_{i=1}^n \frac{dz_i}{z_i}$. Then

$$(36) \quad SH^p(T) \cong \mathbb{Z}[N] \otimes \Lambda^p M \cong \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \otimes \Lambda^p [x_1^{\vee}, \dots, x_n^{\vee}]$$

where x_i and x_i^{\vee} represent dual bases of N and M respectively.

This computation is a special case of the symplectic homology of cotangent bundles [1, 21, 25]. Since we are using the cohomological convention, and the convention that the canonical map $H^*(M) \rightarrow SH^*(M)$ has degree zero, the isomorphism relating the symplectic cohomology to the loop space of a spin manifold Q is

$$(37) \quad SH^{n-*}(T^* Q) \cong H_*(\mathcal{L}Q)$$

Note that (37) “implicitly fixes all the conventions used in the present paper (homology versus cohomology, the grading, and the inclusion of non-contractible loops).”

The subspace $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \Lambda^*[x_1^{\vee}, \dots, x_n^{\vee}]$ is the cohomology of the component of $\mathcal{L}T^n$ consisting of loops representing a certain class $(a_1, \dots, a_n) \in N = H_1(T^n, \mathbb{Z})$, since this component is homotopy equivalent to T^n . The isomorphism (37) maps this subspace in the manner of Poincaré duality to the homology of the loop space.

For a manifold M , string topology shows [10] that the space $\mathcal{L}M$ has a product given by composing families of loops when they are incident, and a BV operator given by spinning the parametrization of the loops. The isomorphism (37) identifies these structures as well [2].

We will briefly explain how this computation can be done from a symplectic viewpoint, previewing the method used for general log Calabi–Yau surfaces. The complex torus can in some way serve as a local model for the general computation. We will describe in section 4 a general method for finding “nice” Liouville structures on log Calabi–Yau surfaces. Mainly, this amounts to a contact-type hypersurface $\Sigma \subset (\mathbb{C}^{\times})^n$ fibered by Lagrangian tori, such that the Reeb flow acts preserving the tori, and rotating each by some amount. Let $\text{Log} : (\mathbb{C}^{\times})^n \rightarrow \mathbb{R}^n$ be the standard torus fibration given by the logarithm map. Let $S \subset \mathbb{R}^n$ be some large sphere centered at the origin, and let $\Sigma = \text{Log}^{-1}(S)$ be the union of the torus fibers sitting over S . We arrange that Σ is contact-type, with the evident torus fibration $\pi : \Sigma \rightarrow S$, that the Reeb vector field is tangent to the fibers of π , and that the Reeb flow acts on each fiber $\pi^{-1}(s)$ as a linear translation on the torus, say translation by the vector $v(s)$. What is important is that the direction of translation depends on the point in the base. Given the base point s , represent the torus $\pi^{-1}(s)$ as $\mathbb{R}^n / \mathbb{Z}^n$; if the direction of $v(s)$ is rational in this representation, the Reeb flow on $\pi^{-1}(s)$ is periodic (with some period $T(s)$). Now for each such s , and each multiplicity $r \in \mathbb{N}^{\times}$, we have a torus $T_{s,r} \subset \mathcal{L}(\mathbb{C}^{\times})^n$ of periodic orbits lying on $\pi^{-1}(s)$ that wrap a primitive orbit r times. In fact, we can arrange that the pairs (s, r) indexing the tori correspond bijectively to the nonzero elements in $H_1((\mathbb{C}^{\times})^n, \mathbb{Z})$ (under the Hurewicz map $\mathcal{L}M \rightarrow H_1(M, \mathbb{Z})$)

If we use either a Hamiltonian H with linear or quadratic growth on the cylindrical end, these periodic Reeb orbits correspond to tori of Hamiltonian orbits of period 1 (with the exception that the Reeb period must be less than the asymptotic slope in the linear case). These orbits are evidently degenerate since they come continuous families, but a generic time-dependent perturbation of the Hamiltonian near each torus $T_{s,r}$ (possibly different for each s, r) breaks this manifold of orbits into several non-degenerate orbits. The differential counts cylinders, hence can only connect orbits that are homologous, and so orbits corresponding to different tori $T_{s,r}$ are not connected by differentials.

Using the isomorphism with loop-space homology, we see that, within the set of orbits corresponding to a single torus $T_{s,r}$, the Floer cohomology complex computes the cohomology of the component of the loop space containing $T_{s,r}$. As this component is homotopy equivalent to $T_{s,r}$, we identify this cohomology with $H^*(T_{s,r})$. The contractible orbits of H contribute the ordinary cohomology of $(\mathbb{C}^\times)^n$. Another expression for the cohomology of $(\mathbb{C}^\times)^n$ is then

$$(38) \quad SH^*((\mathbb{C}^\times)^n) \cong H^*((\mathbb{C}^\times)^n) \oplus \bigoplus_{s,r} H^*(T_{s,r})$$

This shows us that the degree 0 generators of symplectic cohomology correspond to the fundamental classes of the iterates of the periodic tori.

We can use this computation to draw some conclusions about the structure of symplectic cohomology near a periodic torus. Since all the classes in $H^*(T_{s,r})$ are essential in Floer cohomology, we see that the perturbation of Hamiltonian near each periodic torus must create at least $\binom{n}{k}$ orbits of Conley-Zehnder index k .

The BV operator on symplectic cohomology is identified with the rotation of loops $H_*(\mathcal{L}T) \rightarrow H_{*+1}(\mathcal{L}T)$. Thus the action of Δ on $H^*(T_{s,r})$ is Poincaré dual to the operation of taking a cycle $T_{s,r}$ and rotating the parametrization of the loops. Under the isomorphism $H^*(T_{s,r}) \cong \Lambda^* \mathbb{K}^n$, this corresponds to contraction with the class of the orbit.

The product structure is also straightforward. There are maps

$$(39) \quad H^*(T_{s,r}) \otimes H^*(T_{s',r'}) \rightarrow H^*(T_{s'',r''})$$

that are nonzero precisely when the homology classes a, a', a'' represented by the three tori satisfy $a + a' = a''$, and in this case it is given by a sort of cup product. In particular, if we consider $s = s' = s''$, and $r'' = r + r'$, then the map is nontrivial, and the degree zero component of the target is in the image.

3. LOG CALABI-YAU SURFACES

3.1. Basics. Our main objects of study are log Calabi-Yau pairs (Y, D) with positive, maximal boundary. We define these notions presently. For the reader who finds this terminology perverse we note that this combination of conditions is equivalent to saying that Y is a surface and D is a nodal reduced anticanonical divisor, such that D supports an ample divisor class. Readers who are familiar with these notions may skip this section but should note Lemma 3.1, which is relied on throughout the paper.

Let Y denote a smooth projective surface over the complex numbers. The canonical bundle is denoted Ω_Y^2 , and the canonical divisor class is denoted K_Y . Let D be an effective divisor on Y .

Definition 13. The pair (Y, D) is a *log Calabi-Yau pair* if $K_Y + D$ is a principal divisor, that is, D lies in the anticanonical divisor class. Equivalently, there is an isomorphism $\Omega_Y^2(D) \cong \mathcal{O}_Y$.

In this paper we will usually assume that D is a normal crossings divisor.

Definition 14. An effective divisor D on a surface Y is a *normal crossings divisor* if D is a reduced Cartier divisor, and, writing $D = \sum_i D_i$ with irreducible components D_i , each D_i is a smooth or nodal curve intersecting the other components transversely (so that $(D - D_i)|_{D_i}$ is a reduced divisor on D_i).

There is a restriction on the topology of pair (Y, D) when D is normal crossings. The proof is an exercise in adjunction.

Lemma 3.1. *Let D be a connected normal crossings divisor in a smooth projective surface Y , such that the pair (Y, D) is log Calabi-Yau. Then either*

- (1) D is a smooth genus one curve,
- (2) D is a irreducible nodal curve of arithmetic genus one, or
- (3) D is a sum of smooth rational curves, whose intersection graph is a cycle.

Proof. The arithmetic genus of the possibly nodal curve D is given by the adjunction formula and the assumption $K_Y + D \sim 0$,

$$(40) \quad p_a(D) = \frac{(K_Y + D) \cdot D}{2} + 1 = 1.$$

If D is irreducible, it falls under one of the first two cases.

Suppose that $D = \sum_{i \in I} D_i$ is reducible with components D_i . Let $\Gamma(D)$ be the intersection graph of D . The vertex set I is the index set for the components, each vertex is labeled by the arithmetic genus of the component, and we draw as many edges between two vertices as there are intersections between the corresponding components. Our assumption is that $\Gamma(D)$ is connected and has at least two vertices.

We claim that each vertex of $\Gamma(D)$ has valence at least two. Suppose $\Gamma(D)$ has a vertex of valence one, say k . Then $(D - D_k) \cdot D_k = 1$. Using $-K_Y \sim D$,

$$(41) \quad -K_Y \cdot D_k = (D - D_k) \cdot D_k + D_k \cdot D_k = 1 + D_k \cdot D_k.$$

Reducing modulo 2,

$$(42) \quad K_Y \cdot D_k \not\equiv D_k \cdot D_k \pmod{2},$$

which is impossible. Thus every vertex of $\Gamma(D)$ has valence at least two.

Since every vertex as valence at least two $\Gamma(D)$ cannot be a tree. Therefore it contains a cycle. By genus considerations there can only be one cycle, and every component of D is rational. Since $\Gamma(D)$ is connected and contains only one homological cycle, $\chi(\Gamma(D)) = 0$. Thus the numbers of vertices and edges are equal. Since each vertex has valence at least two, and $\sum_{i \in I} \frac{1}{2}\text{valence}(i) = \#\text{edges}$, each vertex has valence exactly two. Thus $\Gamma(D)$ is a cycle. \square

To see that the assumption that D is connected is necessary, consider $Y = E \times \mathbb{P}^1$ where E is an elliptic curve, with $D = E \times \{0\} \cup E \times \{\infty\}$.

Definition 15. A log Calabi–Yau pair (Y, D) satisfying the hypotheses of Lemma 3.1 is said to have *maximal boundary*, or is called a *Looijenga pair*, if it falls under cases 2 or 3 of the conclusion.

For the grading on symplectic cohomology of $U = Y \setminus D$, it is important to actually specify the trivialization of Ω_U^2 . There is a preferred trivialization, given by a meromorphic two-form on Y that is non-vanishing and holomorphic on U with simple poles along D . We have the following proposition, that in particular shows that the homotopy class of the trivialization does not depend on the choice of compactification.

Lemma 3.2. *Let (Y, D) be a log Calabi–Yau pair with maximal boundary. The complement $U = Y \setminus D$ carries a non-vanishing holomorphic two-form Ω , characterized up to a scalar multiple by the property that Ω has simple poles along D . If (Y, D) and (Y', D') are two log Calabi–Yau compactifications of a given U , then the corresponding two-forms Ω and Ω' differ by a scalar multiple.*

Proof. The second assertion implies the first, so it suffices to consider two pairs (Y, D) and (Y', D') such that $Y \setminus D = U = Y' \setminus D'$. Let Ω (resp. Ω') be any meromorphic form on Y (resp. Y') that is non-vanishing and holomorphic on U and has simple poles along D (resp. D'). There is a birational map $p : Y \dashrightarrow Y'$ that is the identity on U . The pull-back $p^*\Omega'$ is a meromorphic form on Y , that is non-vanishing and holomorphic on U . Thus the ratio $f = \Omega/p^*\Omega'$ is a rational function on Y , whose divisor of zeros and poles is contained in D .

When Y and Y' are the same, the condition that both Ω and Ω' have the same divisor of poles implies that f has no zeros or poles, and hence is constant.

If Y and Y' are distinct, we make use of another characterization of Ω , namely that it has nonzero residue at any node of D . This residue is integral of Ω a small torus linking the node. Since Ω is a closed form on U (being a holomorphic top form), we can integrate over any homologous torus, and since the boundary divisor is a cycle of rational curves, the tori at each of the nodes are homologous to each other. If we were to multiply Ω by a rational function f that vanishes along a component of D , then by integrating over a small torus centered on that component we find that the residue of $f\Omega$ would vanish.

Returning to the forms Ω and $p^*\Omega'$, we see that both have a nonzero residue, as this residue may be computed inside U , the locus on which the birational map p is an isomorphism. Thus the ratio f is constant. \square

Definition 16. A reduced divisor $D = \bigcup_{i=1}^n D_i$ supports an ample divisor if some linear combination of the irreducible components $A = \sum_{i=1}^n a_i D_i$ is ample.

Lemma 3.3. Let D be a connected divisor in a projective surface Y . If D supports an ample divisor, then D supports an ample divisor $A = \sum_{i=1}^n a_i D_i$ where all coefficients a_i are strictly positive.

Proof. This proof is drawn from [13, §2.4] via [27]. Consider the set

$$(43) \quad \mathcal{S} = \left\{ A = \sum_{i \in I} a_i D_i \mid I \subset \{1, \dots, n\}, (\forall i \in I)(a_i > 0 \text{ and } A \cdot D_i > 0) \right\}$$

In words, \mathcal{S} is the set of effective divisors, supported on D , that have positive intersection with any irreducible component of their support. The conclusion follows once we know that \mathcal{S} contains a divisor B whose support is all of D . By definition, B contains every D_i with a strictly positive coefficient a_i . The Nakai–Moishezon criterion implies that B is ample: First, by definition $B \cdot D_i > 0$ for every irreducible component of D . Then we also see that $B^2 = \sum_{i=1}^n a_i(B \cdot D_i) > 0$. Lastly, if we consider an irreducible curve C that is not a component of D , we have $C \cdot D_i \geq 0$ for all i . The fact that D supports an ample divisor implies that C is not disjoint from D , so $C \cdot D_i > 0$ for some i . Thus $B \cdot C > 0$.

It remains to show that \mathcal{S} contains an element whose support is all of D . First we show that \mathcal{S} is not empty. Start with some ample divisor $A = \sum_i a_i D_i$ supported on D , and write $A = P - N$, where P and N are effective and have no components in common. Let D_i be contained in the support of P . As $A \cdot D_i > 0$, we have $P \cdot D_i > N \cdot D_i$. Since D_i is not contained in the support of N , we have $N \cdot D_i \geq 0$, and so $P \cdot D_i > 0$. Thus $P \in \mathcal{S}$.

Now we must add the other components of D while staying in \mathcal{S} . Let D_j be an irreducible component of D that is not contained in P , but which does intersect P non-trivially. Then $mP + D_j \in \mathcal{S}$ for $m \gg 0$. Indeed, $(mP + D_j) \cdot D_j > 0$ as long as $m > -D_j^2/P \cdot D_j$ (the denominator is greater than zero by assumption that D_j intersects P and is not contained in it).

Because we assumed that D is connected, we may iterate the previous step to add each time an irreducible component of D that is not contained in the support but which intersects it non-trivially. Thus \mathcal{S} contains an element whose support is all of D . \square

3.2. Examples. Observe that a pair (Y, D) is a log Calabi–Yau pair with maximal boundary and D ample if and only if Y is a del Pezzo surface and D is an nodal reduced anticanonical divisor. However, the weaker assumption that D merely supports an ample divisor includes infinitely many more types of surfaces. For example, we can take $Y = \mathbb{P}^2$ blown up any number of times, as long as these blowups all lie on a single conic. We let $D = Q \cup L$ be the union of the proper transform of that conic Q with some line L . This D is anticanonical and nodal, and $mL + Q$ is ample for $m \gg 0$.

Here we list some examples of log Calabi–Yau pairs.

3.2.1. The projective plane. Let $Y = \mathbb{P}^2$. As $\Omega_{\mathbb{P}^2}^n \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, any cubic curve will serve for D . There are essentially four possibilities. In going from each case to the next, we smooth a node of D . This changes the complement U by adding a 2-handle.

- (1) D is the union of three lines in general position. Then $U = Y \setminus D \cong (\mathbb{C}^\times)^2$, and the Betti numbers are $b_1 = 2, b_2 = 1$.
- (2) D is the union of a conic and a line in general position. The Betti numbers of U are $b_1 = 1, b_2 = 1$. Floer cohomology for Lagrangian submanifolds in U was studied in [19].
- (3) D is a nodal cubic curve. The Betti numbers of U are $b_1 = 0, b_2 = 1$.
- (4) D is a smooth cubic curve. The pair (Y, D) does not have maximal boundary. The Betti numbers of U are $b_1 = 0, b_2 = 2$.

3.2.2. A cubic surface. Let Y be a smooth cubic surface in \mathbb{P}^3 . As is well-known, Y contains 27 lines, and it is possible to choose three of them intersecting in a 3-cycle so that their sum is an anticanonical divisor. To see this, realize Y as the projective plane blown up in six general points p_1, \dots, p_6 , giving six exceptional curves E_1, \dots, E_6 . Let L_{ij} denote the proper transform of the line through p_i and p_j (there are 15 of these). Let C_k denote the proper transform of the conic through five of the points, all except p_k (there are six of these). The curves E_i, L_{ij}, C_k are the 27 (-1) -curves that are mapped to lines by the anticanonical embedding $Y \rightarrow \mathbb{P}^3$.

For two indices a and b , consider the configuration L_{ab}, E_b, C_a . This means the line through p_a, p_b , the exceptional curve over p_b , and the conic that does not contain p_a . Clearly L_{ab} and E_b intersect over p_b , while C_a and E_b intersect since C_a passes over p_b . Also, L_{ab} and C_a intersect since their projections to \mathbb{P}^2 intersect in two points: one point is p_b , and the other point is none of the p_i (which are assumed to be in general position), and this latter intersection point persists in the blow-up.

The divisor $D = L_{ab} + E_b + C_a$ is anticanonical and very ample.

3.2.3. The degree 5 del Pezzo surface. Let Y be the (unique) degree 5 del Pezzo surface, realized as the blow-up of \mathbb{P}^2 at 4 general points p_1, p_2, p_3, p_4 . There are 10 (-1) -curves in Y , namely the 4 exceptional curves E_i coming from the blow-ups, and $\binom{4}{2} = 6$ proper transforms of the lines passing through two of the points L_{ij} . To get an anticanonical divisor, choose a partition of the set $\{1, 2, 3, 4\}$ into $\{i, j\}$ and $\{k, \ell\}$ (there are 12 such choices). Then take $D = L_{ki} + E_i + L_{ij} + E_j + L_{j\ell}$. Thus D is a 5-cycle of (-1) -curves.

3.2.4. “Punctured” A_n Milnor fibers. Let $V = \{x^2 + y^2 + z^{n+1} = 1\} \subset \mathbb{C}^3$ be the Milnor fiber of the two-dimensional A_n singularity. It is possible to compactify V by adding two rational curves [12, §7.1]. Start with \mathbb{P}^2 , with homogeneous coordinates $[x : y : z]$. Blow up the $n+1$ points $[\xi_k : 0 : 1]$ along the x -axis, where $\xi_k = \exp(2\pi i k / (n+1))$, and call the result Y . Let P_t be pencil on Y that is the preimage of the pencil of lines through $[0 : 1 : 0]$. In an affine chart these are depicted as the lines parallel to the y -axis. The line at infinity $\{z = 0\}$ is a fiber of this pencil, P_∞ . The pencil P_t on Y has $n+1$ singular fibers, where the line passes through a blown-up point. Let C denote the proper transform of the x -axis; it is a section of the pencil and passes through all of the exceptional curves. The complement $Y \setminus (P_\infty \cup C)$ is isomorphic as a complex manifold to the Milnor fiber V [12, Lemma 7.1].

Although V satisfies $c_1(V) = 0$, it is not log Calabi–Yau in the sense of this paper. The anticanonical class of Y is

$$(44) \quad -K_Y \sim 3H - \sum_{i=1}^n E_i \sim 2P_\infty + C$$

where H denotes the pull-back of the hyperplane class on \mathbb{P}^2 . The issue is the coefficient of 2 in front of P_∞ , which means that a holomorphic volume form on V will have a pole of order 2 along P_∞ .

We can get something that falls into our setting by removing another smooth fiber of the pencil, say P_0 . Write $U = Y \setminus (P_0 \cup P_\infty \cup C)$. We call U the punctured A_n Milnor fiber, since we puncture the line parametrizing the pencil on V . As $-K_Y \sim P_0 + P_\infty + C$, this is a log Calabi-Yau surface.

The self-intersections are $P_0^2 = 1, P_\infty^2 = 1, C^2 = 1 - n$. The compact surface Y is not Fano unless $n \leq 2$, since $-K_Y \cdot C = 3 - n$. Nevertheless, the divisor $P_0 \cup P_\infty \cup C$ supports an ample divisor. Indeed, $aP_0 + bP_\infty + C$ is ample as long as $a > 0, b > 0$, and $a+b > n-1$.

Remark 1. The point of view that U is the form of the Milnor fiber that is “truly Calabi-Yau” comes from the Strominger–Yau–Zaslow (SYZ) picture and is discussed in [4, §9.2]. This lines up well with [16], which also used the SYZ picture (in the form of the Gross–Siebert program) as its starting point.

4. CONSTRUCTION OF THE LIOUVILLE DOMAIN

In this section we will construct a Liouville domain associated to a log Calabi-Yau pair (Y, D) where D is positive. In order to obtain symplectic forms, we use Lemma 3.3, and choose an ample divisor $A = \sum a_i D_i$ supported on D , such that each coefficient a_i is strictly positive. The Liouville domain we construct is the symplectic model for the complement $U = Y \setminus D$. Since we are ultimately interested in symplectic homology, and symplectic homology is an invariant of Liouville deformation, we are free to take a particular representative of the Liouville deformation class that has convenient properties. This idea was used by Seidel [22] and Mark McLean [18] to understand the growth rate of symplectic cohomology. In fact, the first four steps of the construction follow [22, §4] very closely, though at some points we extract more precise information for our particular cases. The fifth step is new, and highlights an interesting property of contact hypersurfaces in U .

4.1. Basic Liouville structure on an affine variety. Let Y be a smooth projective variety with a positive divisor D , and let $A = \sum a_i D_i$ be a strictly positive combination of components that is ample. Then there is a holomorphic line bundle $\mathcal{L} \rightarrow Y$ and a section $s \in H^0(Y, \mathcal{L})$ such that $A = s^{-1}(0)$. The line bundle \mathcal{L} admits a Hermitian metric $\|\cdot\|$ such that, if F is the curvature of the unique connection compatible with the metric and the holomorphic structure, then $\omega = 2iF$ is a Kähler form. On the complement $U = Y \setminus D$, the function $\phi = -\log \|s\|$ is a Kähler potential since

$$(45) \quad 2iF|_U = dd^c(-\log \|s\|)$$

Thus the symplectic form ω on U is exact and $\lambda = d^c\phi$ is a primitive. Here $d^c = -i(\partial - \bar{\partial})$, and for a function f this means $d^c f = -df \circ J$.

The function ϕ is clearly proper and bounded below. A simple lemma [22, Lemma 4.3] shows that the set of critical points of ϕ is compact when D has normal crossings, so by choosing a sufficiently large regular value C , we find that $\overline{U} = \phi^{-1}(-\infty, C]$ is a compact subset containing all of the topology of U .

The Liouville vector field is defined by the condition $\iota_Z \omega = \lambda$. Thus

$$(46) \quad 0 \leq g(Z, Z) = \omega(Z, JZ) = \lambda(JZ) = -d\phi \circ J(JZ) = d\phi(Z)$$

and equality can only hold when $Z = 0$, whence $\lambda = 0$, whence $d\phi = 0$. Thus Z points strictly outwards along $\partial\overline{U} = \phi^{-1}(C)$.

Thus \overline{U} equipped with the structures ω, λ, Z is a Liouville domain.

4.2. Refinements of the basic construction. We now assume that in addition to being positive, D is anticanonical, so that (Y, D) is a log Calabi–Yau pair. We also assume that the pair has maximal boundary, which means that D is normal crossing with at least one node. By Lemma 3.1, D is either isomorphic to the irreducible nodal genus one curve or it is a cycle of rational curves. In a sense, the former case is subsumed into the latter if we regard the irreducible nodal genus one curve as a rational curve intersecting itself, and draw the intersection graph as one vertex with one self-edge. In what follows we consider the case where the irreducible components are smooth rational curves, but with a certain amount of notational inconvenience all the arguments go through for the irreducible case.

To begin with we start with a basic Liouville structure ω, λ, Z on U as in the previous section.

4.2.1. Step 1: constructing local torus actions along the divisor. This step is basically the same as in [22], but we get a little more structure along the smooth parts of the divisors.

Let us write $D = \bigcup_{i=1}^r D_i$, where each irreducible component D_i is a smooth rational curve, and the components are ordered cyclically according to some chosen orientation of the intersection graph. So $D_i \cap D_{i+1} = 1$. The first thing to do is to make consecutive divisor symplectically orthogonal. According to [22, §4, Step 1] we may choose the metric on \mathcal{L} so that in local coordinates (z_1, z_2) near $D_i \cap D_{i+1}$, the divisor is $D = \{z_1 z_2 = 0\}$ and the Kähler form is standard. This neighborhood is a toric manifold with moment map $m(z_1, z_2) = \frac{1}{2}(|z_1^2|, |z_2^2|)$.

This moment map can actually be extended over tubular neighborhoods of D_i and D_{i+1} . Let us consider D_i . By the symplectic tubular neighborhood theorem, a disk bundle $\mathbb{D}(N_i)$ contained in the normal bundle N_i to D_i in Y is symplectomorphic to a tubular neighborhood of D_i . There is a Hamiltonian circle action on D_i that fixes the points $D_{i-1} \cap D_i$ and $D_i \cap D_{i+1}$ where D_i intersects the rest of the divisor. Choosing some S^1 -equivariant structure on N_i , we can lift the circle action to $\mathbb{D}(N_i)$. There is also a Hamiltonian circle action on $\mathbb{D}(N_i)$ that rotates the fibers. Thus $\mathbb{D}(N_i)$ is a toric manifold. So constructed, we find that the “base” S^1 -action rotates the fibers over the fixed points by some (possibly different) amounts. Nevertheless, at each fixed point, we may find a chart (z_1, z_2) such that $D_i = \{z_1 = 0\}$ and the fiber over the fixed point is $\{z_2 = 0\}$. Then we may take a basis of the Lie algebra of T^2 such that the moment map is again standard. Choosing the tubular neighborhood so that D_{i-1} and D_{i+1} correspond to the normal lines over the fixed points, we are done.

The outcome is that a tubular neighborhood of D is locally toric, and in fact, if we excise a ball around one of the nodes, there is a neighborhood of the rest of D which admits a Hamiltonian T^2 -action. See Figure 1. We observe that there the T^2 -orbits in the neighborhood of the divisor form a Lagrangian torus fibration on this set.

4.2.2. Step 2: making the Liouville form symmetric near the crossings. This step is identical to [22, §4, Step 2] so we will be brief. The outcome is that there is a smooth function k on Y such that $\lambda' = \lambda + dk$ is T^2 -invariant near each of the crossings, and that the sublevel set $\overline{U} = \phi^{-1}(\infty, C]$ (for a large regular value C) equipped with the restriction of λ' is still a Liouville domain. This Liouville structure is deformation equivalent to the original structure.

4.2.3. Step 3: making the boundary torus invariant. The goal is to construct an exhausting family of Liouville domains $\overline{U}' \subset U$ such that the boundary $\partial\overline{U}'$ is invariant under the local torus actions. This condition tells us what to do: in a portion of the neighborhood of the divisor where the moment map is defined, let S be a path that goes very close to the boundary of the moment map image. More precisely, we take the boundary of the moment map image, push it off into the interior of this image, and then round the corners in the simplest way. See Figure 2. Then let Σ be the union of the torus orbits over this path.

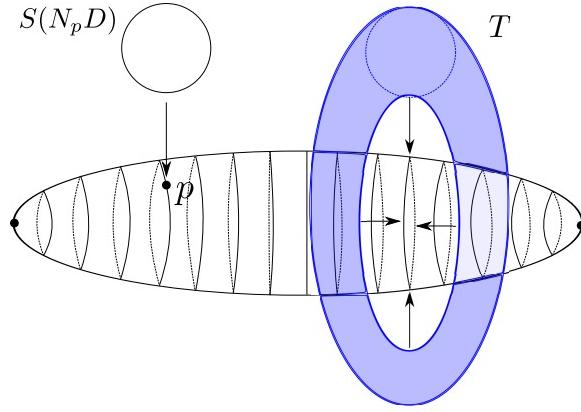


FIGURE 1. Tori in the normal bundle to a rational curve.

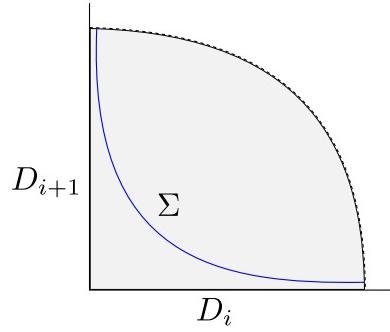


FIGURE 2. Moment map picture for the contact hypersurface near a node.

By looking in several charts we may define Σ as a closed real three dimensional manifold contained in a neighborhood of the divisor, which has the topology of a T^2 -bundle over the circle S .

We take \overline{U}' to be the inside of the real hypersurface Σ , namely the side not containing D . The arguments of [22, §4, Step 3] apply to this hypersurface Σ to show that if the path S is taken close enough to the boundary, the Liouville vector field Z' from Step 2 points outward along Σ . We also find that $(\overline{U}, \lambda'|_{\overline{U}})$ and $(\overline{U}', \lambda'|_{\overline{U}'})$ are Liouville isomorphic.

4.2.4. Step 4: making the contact form torus invariant. This is the same as in [22] but we use the T^2 -action instead of just the S^1 -action. Near the crossings, λ' is already T^2 -invariant, as is Σ , so $\lambda'|_{\Sigma}$ is a T^2 -invariant contact form. Away from the crossings, we find that Z' points outwards all along the torus fibers of Σ , so by averaging over the T^2 -action we obtain a Liouville form λ'' defined in a neighborhood of Σ that is T^2 -invariant, and whose dual vector field Z'' points outwards along Σ . The desired contact form is $\alpha = \lambda''|_{\Sigma}$. The contact structure is isomorphic to the one induced by λ' , and so there is a isomorphic Liouville structure on \overline{U}' whose boundary has the contact form α .

4.3. Reeb dynamics and Liouville classes. Now we investigate further the contact boundary (Σ, α) that is the output of Step 4. Recall that Σ is a torus bundle over a

circle S .

$$(47) \quad \begin{array}{ccc} T^2 & \longrightarrow & \Sigma \\ & & \downarrow \pi \\ & & S \end{array}$$

and the contact form α is invariant under the local T^2 -action. Note that Σ is not a principal T^2 -bundle since there is no global T^2 -action, and indeed, the structural group of the fiber bundle is not T^2 -translations but rather diffeomorphisms of the torus.

Let $I \subset S$ be some interval, and consider $\Sigma|_I = \pi^{-1}(I)$. Then $\Sigma|_I$ has a T^2 action, and by choosing a section $\sigma|_I : I \rightarrow \Sigma|_I$ of the fibration, we obtain an equivariant diffeomorphism

$$(48) \quad \Sigma|_I \cong T^2 \times I$$

Introduce coordinates (θ_1, θ_2, s) on $T^2 \times I$, where (θ_1, θ_2) are 2π -periodic coordinates on the fiber, and $s \in I$ is a coordinate on the base. Since the contact form α is a T^2 -invariant one-form, in this coordinate system it can be written

$$(49) \quad \alpha = f(s) d\theta_1 + g(s) d\theta_2 + h(s) ds$$

for some functions f, g, h that depend on s but not on the angular coordinates. We compute

$$(50) \quad d\alpha = f'(s) ds \wedge d\theta_1 + g'(s) ds \wedge d\theta_2 = ds \wedge \frac{\partial \alpha}{\partial s}$$

Where the partial derivative $\partial \alpha / \partial s$ is taken with respect to the coframe $(d\theta_1, d\theta_2, ds)$. The volume form is

$$(51) \quad \alpha \wedge d\alpha = \alpha \wedge ds \wedge \frac{\partial \alpha}{\partial s} = -ds \wedge \alpha \wedge \frac{\partial \alpha}{\partial s} = - \begin{vmatrix} f(s) & g(s) \\ f'(s) & g'(s) \end{vmatrix} ds \wedge d\theta_1 \wedge d\theta_2$$

4.3.1. An orientation convention. The orientation on Σ is induced by from the filling \overline{U}' by the “outward normal first” convention, which is the same as the orientation induced by the volume form $\alpha \wedge d\alpha$. Since we have a fibration structure where the total space is canonically oriented, an orientation of the base is equivalent to an orientation of the fiber. The convention is that for $F \rightarrow E \rightarrow B$ a fibration, $\Lambda^{\max} T^* E \cong \Lambda^{\max} T^* B \otimes \Lambda^{\max} T^* F$. Assuming that the base is oriented by the form ds , we see that the fiber is oriented by the volume form

$$(52) \quad - \begin{vmatrix} f(s) & g(s) \\ f'(s) & g'(s) \end{vmatrix} d\theta_1 \wedge d\theta_2$$

By switching the roles of f, g and θ_1, θ_2 if necessary, we may assume that the determinant is negative, so that $d\theta_1 \wedge d\theta_2$ is a positive volume form. A change in the orientation of the base also leads to a switch of this form.

4.3.2. The Reeb vector field. The Reeb vector field R is determined by the conditions $\iota_R d\alpha = 0$ and $\alpha(R) = 1$. Expanding R in the coordinate frame,

$$(53) \quad R = R_1 \partial_{\theta_1} + R_2 \partial_{\theta_2} + R_s \partial_s$$

We find that

$$(54) \quad \begin{aligned} R_s &= 0 \\ f'R_1 + g'R_2 &= 0 \\ fR_1 + gR_2 &= 1 \end{aligned}$$

Observe that R is vertical, that is, tangent to the fibers of π . Since the Reeb vector field spans the characteristic foliation Σ , we see that this foliation is tangent to the torus fibers, and on each torus fiber it consists of lines of some (rational or irrational) slope. The slope of this foliation varies as the fiber moves. See Figure 3.

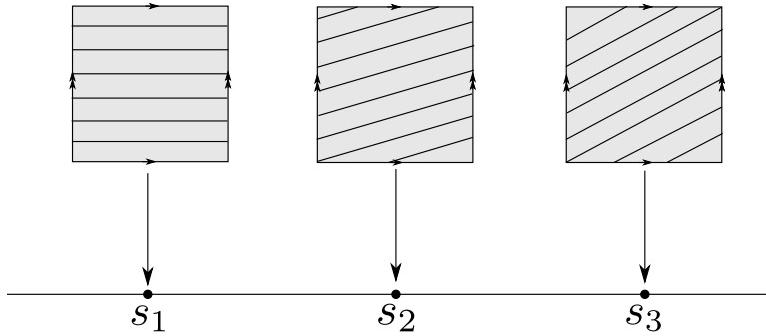


FIGURE 3. The characteristic foliation on various torus fibers.

4.3.3. *The Liouville class.* Now consider the torus fibers $T^2 \times \{s\}$. By equation (50) we see that $d\alpha|_{T^2 \times \{s\}} = 0$, that is, each torus is pre-Lagrangian. Thus we have the Liouville class

$$(55) \quad A(s) = [\alpha|_{T^2 \times \{s\}}] = f(s)[d\theta_1] + g(s)[d\theta_2] \in H^1(T^2 \times \{s\}, \mathbb{R})$$

The cohomology-valued function A may be regarded as a section of the rank two vector bundle

$$(56) \quad R^1\pi_*(\underline{\mathbb{R}}) \rightarrow S$$

This vector bundle has a flat Gauss-Manin connection ∇ with monodromy $\mu \in \mathrm{SL}_2(\mathbb{Z})$. In particular, the expression ∇A defines a one-form on S with values in this bundle. The cup product on the fibers then yields an element

$$(57) \quad A \cup \nabla A \in \Omega^1(S, R^2\pi_*(\underline{\mathbb{R}}))$$

Here $R^2\pi_*(\underline{\mathbb{R}})$ is a bundle whose fiber at s is $H^2(T^2 \times \{s\}, \mathbb{R}) \cong \mathbb{R}$, and this vector bundle is trivialisable using the orientation convention discussed above.

4.3.4. *Legendrian sections and elimination of $h(s)$.* We observe that the condition that α is contact is equivalent to the non-vanishing of the determinant in the last expression of equation (51). Since this determinant does not involve $h(s)$, we may change it arbitrarily while preserving the contact condition, and, by Gray's theorem, the isomorphism class of the contact structure. The following lemma will be used to eliminate the $h(s) ds$ term by a change of coordinates.

Lemma 4.1. *The fibration $\pi : \Sigma \rightarrow S$ admits a section $\sigma : S \rightarrow \Sigma$ whose image is a Legendrian circle in Σ .*

Proof. Choose a global smooth section $\sigma_0 : S \rightarrow \Sigma$ (it exists because the base is a circle and the fiber is connected). Letting s denote a coordinate on S and ∂_s the coordinate vector field, consider $\alpha(T\sigma_0(\partial_s))$, which is a function on S . Then

$$(58) \quad \alpha_t = \alpha - t \cdot \pi^*[\alpha(T\sigma_0(\partial_s))] ds$$

is a family of contact forms starting from $\alpha_0 = \alpha$ and such that $\sigma_0(S)$ is a Legendrian circle for α_1 . Denote the corresponding contact structures by ξ_t . By Gray's theorem, there is an isotopy $\psi_t : \Sigma \rightarrow \Sigma$ such that $T\psi_t(\xi_0) = \xi_t$. Thus $\psi_1^{-1}(\sigma_0(S))$ is a Legendrian circle for the contact form α_0 .

To see that this circle is still a section of π , we look closer at the application of Moser's trick that is used to construct the isotopy [15, p. 60]. The isotopy ψ_t is the flow of the time-dependent vector field $X_t \in \xi_t$ which is chosen to satisfy

$$(59) \quad \dot{\alpha}_t + \iota_{X_t} d\alpha_t = \mu_t \alpha_t$$

where

$$(60) \quad \mu_t = \dot{\alpha}_t(R_t)$$

Since $d\alpha_t$ is independent of t , we find that all the Reeb vector fields R_t are vertical, and since the deformation term is pulled back from the base, they are actually all the same. Since $\dot{\alpha}_t$ is pulled back from S and $R_t = R$ is vertical, we see that $\mu_t \equiv 0$. The first equation becomes

$$(61) \quad \pi^*[\alpha(T\sigma_0(\partial_s))ds] + \iota_{X_t}d\alpha = 0$$

Due to the specific form of $d\alpha$ in equation (50), see that X_t is vertical, and is invariant under the local T^2 -actions at all times. We conclude that the isotopy ψ_t acts vertically (preserving the fibers) and equivariantly for the local T^2 -actions. Thus $\sigma = \psi_1^{-1} \circ \sigma_0$ is a Legendrian section. \square

From now on we will pick some Legendrian section $\sigma : S \rightarrow \Sigma$. Restricting σ to some interval $I \subset S$, we obtain an equivariant diffeomorphism $\Sigma|_I \rightarrow T^2 \times I$ such that α has the form

$$(62) \quad \alpha = f(s)d\theta_1 + g(s)d\theta_2$$

The $h(s)ds$ term from equation (49) is not present since the lines $\{(\theta_1, \theta_2)\} \times I$ are now Legendrian, and so $\alpha(\partial_s) = 0$. We now observe that such a contact form α is entirely determined by the Liouville class $A(s)$ (55).

4.3.5. Properties of the Liouville class. Observe that for a (pre-)Lagrangian torus $L \subset \Sigma \subset \overline{U}'$, we have

$$(63) \quad [\alpha|_L] = [\lambda''|_L] = [(\iota(Z'')\omega)|_L]$$

where λ'', Z'' are from the Liouville structure that is the output of Step 4. The properties of the Liouville class we need to use are just a translation of the properties of Z'' that are ensured by the construction in section 4.2. However, the Liouville class is easier to compute since, being a cohomology-level object, it is more stable under deformations.

Proposition 4.2. *Let $L \subset \Sigma \subset \overline{U}'$ be a Lagrangian torus near the boundary divisor, which is furthermore an orbit of the local torus actions. Then the Liouville class $[\lambda''|_L]$ for the Liouville structure coming from Step 4 is equal to the Liouville class $[\lambda|_L]$ for the Liouville structure after Step 1.*

Proof. Step 2 does not alter the Liouville class because it changes the one-form by a globally exact form:

$$(64) \quad \lambda' = \lambda + dk$$

Step 3 involves no change in the Liouville structure. Step 4 changes the Liouville structure by averaging λ' over the local torus actions. Since L is assumed to be an orbit of the torus action, we find

$$(65) \quad \lambda'' = \int_{T^2} ((t_1, t_2)^*\lambda') \frac{dt_1 dt_2}{(2\pi)^2}$$

but all the forms $(t_1, t_2)^*\lambda'$ in the integrand are cohomologous since they are isotopic. So at the cohomology level we are just averaging a constant function. \square

Because of the stability expressed in proposition 4.2, we can compute the Liouville classes of our tori using the Liouville structure coming from Step 1. Recall that this Liouville structure comes from Kähler geometry: $D = s^{-1}(0)$ for some section s of the ample line bundle \mathcal{L} , we have chosen $\|\cdot\|$ an Hermitian metric on \mathcal{L} such that $\phi = -\log \|s\|$ is the Kähler potential, and in Step 1 we ensured that the Kähler form is standard near the crossings.

Now we describe this in an analytic coordinate chart $V \subset Y$, possibly containing part of the divisor D . Over V we also choose a holomorphic trivialization $\mathcal{L}|_V \cong \mathcal{O}|_V$. With respect to this trivialization, the holomorphic section s becomes a holomorphic function that we also denote by s . The Hermitian metric $\|\cdot\|$ is, at each point, a positive multiple of the absolute value norm on \mathcal{O} , so $\|s\| = e^\psi |s|$ for some function $\psi : V \rightarrow \mathbb{R}$. Thus

$$(66) \quad \phi = -\log \|s\| = -\log e^\psi |s| = -\psi - \log |s|$$

Now since s is holomorphic and vanishes on D , we find that $-\log |s|$ is a function which is discontinuous along D , and $dd^c(-\log |s|) = 0$ the complement of D (that is, $-\log |s|$ is pluriharmonic outside of D). If we interpret the formula in terms of weak derivatives, $dd^c(-\log |s|)$ is a current supported along D (the Poincaré–Lelong formula). Thus, on the complement of D , $-\psi$ is another Kähler potential for the same form.

However, the term $-\log |s|$ contributes greatly to the Liouville one-form $\lambda = d^c \phi$. Consider the case near a crossing. We may choose the holomorphic coordinate chart V and the holomorphic trivialization of \mathcal{L} so that $s = z_1^{a_1} z_2^{a_2}$ in local coordinates. If $z_j = \exp(\rho_j + i\theta_j)$,

$$(67) \quad d^c(\log |z_j|) = d\theta_j$$

Thus

$$(68) \quad \lambda = d^c \phi = -d^c \psi - a_1 d\theta_1 - a_2 d\theta_2$$

The term $-d^c \psi$ extends continuously across the divisor D , while the other terms do not. Thus we may write, where $z_j = x_j + iy_j$,

$$(69) \quad -d^c \psi = \beta_1 dx_1 + \beta_2 dy_1 + \beta_3 dx_2 + \beta_4 dy_2$$

where the coefficients $\beta_1, \beta_2, \beta_3, \beta_4$ are bounded continuous functions on the chart V .

Now suppose that $i : T^2 \rightarrow V$ is the embedding of a crossing torus $\{|z_1| = \epsilon_1, |z_2| = \epsilon_2\}$.

$$(70) \quad i(\theta_1, \theta_2) = (\epsilon_1 \cos \theta_1, \epsilon_1 \sin \theta_1, \epsilon_2 \cos \theta_2, \epsilon_2 \sin \theta_2)$$

Then clearly $i^* dx_j$ and $i^* dy_j$ are both in the class $O(\epsilon_j) d\theta_j$ (big-O notation). Thus

$$(71) \quad i^*(-d^c \phi) = O(\epsilon_1) d\theta_1 + O(\epsilon_2) d\theta_2$$

and the Liouville class is

$$(72) \quad [\lambda|_{T^2}] = (-a_1 + O(\epsilon_1)) [d\theta_1] + (-a_2 + O(\epsilon_2)) [d\theta_2]$$

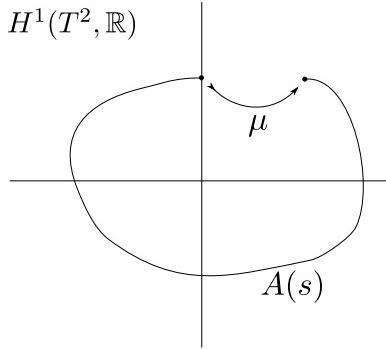
A similar analysis works along the smooth parts of the divisors. The smooth part of each divisor is complex analytically a \mathbb{C}^\times , and we can take an analytic coordinate z_B there (B stands for base). We restrict z_B to lie in some large annulus A in order to avoid going right up to the nodes. The normal bundle of D restricted to A is then holomorphically trivial, and we let z_F be a coordinate on the fibers (F stands for fiber). We may write $s = z_F^a$.

$$(73) \quad \lambda = d^c(-\log \|s\|) = -d^c \psi - d^c(\log |z_F^a|) = -d^c \psi - a d\theta_F$$

Where $d\theta_F$ is the angular one form on the fibers of the normal bundle. The tori in question are not necessarily standard in these coordinates, but if we take a torus $i : T^2 \rightarrow V$ which is within distance ϵ of the divisor, we have

$$(74) \quad [\lambda|_{T^2}] = (-a + O(\epsilon)) [d\theta_F] + \beta [d\theta_B]$$

where we do not assume any control over β .

FIGURE 4. The Liouville class as a path in $H^1(T^2, \mathbb{R})$.

4.3.6. *Step 5: making the Liouville class “locally convex”.* Using the estimates (72) and (74), we can plot the Liouville class A in the first cohomology of the torus. The class A is a section of the flat line bundle $R^1\pi_*(\mathbb{R}) \rightarrow S$ with monodromy $\mu \in \mathrm{SL}_2(\mathbb{Z})$. Choose a basepoint $s_0 \in S$, and a parametrization $\tau : [0, 1] \rightarrow S$ such that $\tau(0) = \tau(1) = s_0$. The flat connection trivializes the bundle pulled back to $[0, 1]$, and identifies all the fibers of the bundle with a model $H^1(T^2, \mathbb{R})$. With these identifications, the class $A(s) \in H^1(\pi^{-1}(s), \mathbb{R})$ then becomes a path in $H^1(T^2, \mathbb{R})$, which is such that

$$(75) \quad A(\tau(1)) = \mu A(\tau(0))$$

See Figure 4.

The computations leading two the estimates (72) and (74) were done in local coordinate charts giving rise to frames $[d\theta_1], [d\theta_2]$ as in (72) and $[d\theta_F], [d\theta_B]$ as in (74). Using the trivialization we plot these vectors in the model space $H^1(T^2, \mathbb{R})$ (this totals to $4n$ vectors if n is the number of nodes.) The estimates imply a picture of $A(s)$ (at least qualitatively.)

- By (72), near each node, $A(s)$ lies in the third quadrant with respect to the basis $[d\theta_1], [d\theta_2]$ coming from that node.
- By (74), near each divisor, $A(s)$ lies in the lower half plane with respect to the basis $[d\theta_B], [d\theta_F]$. Note that the class of $[d\theta_B]$ can be defined up to sign without choosing a trivialization of the normal bundle to the divisor, while $[d\theta_F]$ is only canonical up to adding a multiple of $[d\theta_B]$. Nevertheless this “lower half plane condition” has the canonical meaning that the integral of $A(s)$ over the normal circle fiber is negative.
- We also need to use the relationships between the different frames. This is explained in the self-contained section 5.
- The contact condition $\alpha \wedge d\alpha > 0$ becomes the condition that $A(s)$ as a path is always rotating clockwise with respect to the origin, once we orient everything as in section 4.3.1.

The general picture is of a path in $H^1(T^2, \mathbb{R})$ that winds around possibly several times, and such that the endpoint is the monodromy image of the starting point. (Figure here)

The condition we wish to ensure is that this path is locally convex with respect to the origin.

Definition 17. Let $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ be a path. Then γ is locally convex with respect to the origin if sufficiently short secant lines of γ lie closer to the origin than γ does. For a C^2 path such that $\gamma \wedge \gamma' \neq 0$, this becomes the differential condition

$$(76) \quad \frac{\gamma' \wedge \gamma''}{\gamma \wedge \gamma'} > 0$$

Since the path $A(s)$ has the property that $A \wedge A' < 0$, we just need to ensure that $A' \wedge A'' < 0$ as well. This can be achieved by a conformal rescaling of A , and hence of the contact form α . Indeed, the above observations show that $A(s)$ is approximately straight along each divisor, goes clockwise around the origin, and makes a right turn by some amount at each node. Since we have assumed that (Y, D) has maximal boundary, there is at least one node where such a turn is made. (Step 5 does not work if D is smooth). We modify $A(s)$ to spread out these right turns along the divisor segments.

This modification is effected by a conformal rescaling of the contact form α by a function that depends only on s , so it does not change the underlying contact structure or destroy the toroidal symmetry. Hence there is an isomorphic Liouville structure on \overline{U}' such whose boundary contact form is exactly the locally convex α .

So as not to completely lose the reader with this somewhat strange definition, we put here the crucial lemma that this modification makes possible.

Lemma 4.3. *Let $\varrho \in H_1(\pi^{-1}(s_0), \mathbb{Z})$ be a class of loops such that*

$$(77) \quad \int_{\varrho} A(s_0) > 0, \quad \int_{\varrho} A'(s_0) = 0.$$

By parallel transport extend to $\varrho \in H_1(\pi^{-1}(s), \mathbb{Z})$ for nearby fibers (s near s_0). If $A(s)$ is locally convex, then the period integral

$$(78) \quad I(s) = \int_{\varrho} A(s)$$

has a non-degenerate local maximum at $s = s_0$.

Before the proof, we observe that by section 4.3.2, the hypothesis is satisfied whenever ϱ is the homology class represented by a closed Reeb orbit (see section 6.1.1).

Proof. First observe that s_0 is a critical point:

$$(79) \quad I'(s_0) = \int_{\varrho} A'(s_0) = 0$$

It remains to show that $I''(s_0) < 0$. The local convexity condition

$$(80) \quad \frac{A'' \wedge A'}{A \wedge A'} < 0$$

means that A and A'' lie on opposite sides of the line spanned by A' . Since \int_{ϱ} is a linear function that vanishes on that line, it must take opposite signs on A and A'' . Since it is positive on A , it must be negative on A'' :

$$(81) \quad I''(s_0) = \int_{\varrho} A''(s_0) \quad \text{is opposite to} \quad \int_{\varrho} A(s_0) > 0$$

□

5. THE AFFINE MANIFOLD

We recall the construction of the affine manifolds from [16], but with an emphasize on how it follows from the topology of the neighborhood of the boundary divisor. This section consists of elementary 4-dimensional topology.

5.1. Initial data. Let D be a cycle of m rational curves. By this we mean a curve with m nodes whose normalization is the disjoint union of m rational curves, and such that the dual intersection complex is a cycle. If $m = 1$ this means a nodal elliptic curve. As a matter of notation, we will index the *nodes* by $i \in \mathbb{Z}/m\mathbb{Z}$, and we will index the rational curves either a single index or a pair:

$$(82) \quad D_i = D_{i,i+1} \text{ connects node } i \text{ to node } i+1.$$

Suppose that $\iota : D \rightarrow Y$ is an embedding of the singular curve D into an algebraic surface Y . If $m > 1$ then each component $D_i \cong \mathbb{P}^1$ is embedded and has a normal bundle of some degree k_i :

$$(83) \quad N_i = \iota^* TY / TD_i \cong \mathcal{O}_{\mathbb{P}^1}(k_i).$$

If $m = 1$ we mean $N_1 = (\iota \circ \nu)^*(TY) / T\mathbb{P}^1$, where $\nu : \mathbb{P}^1 \rightarrow D$ is the normalization of the nodal elliptic curve. The topology of a neighborhood of D in Y is completely determined by the numbers k_i , which are also characterized as the self-intersection numbers in Y :

$$(84) \quad k_i = D_i^2$$

5.2. The local model for nodes. Let $\Delta = \{|z| \leq 1\}$ and $\Delta^* = \Delta \setminus \{0\}$ denote the complex unit disk and its punctured version.

Let us consider the i th node. Here the two divisors $D_{i-1,i}$ and $D_{i,i+1}$ intersect transversely. Let us take a neighborhood V_i and local analytic coordinates $(z_i, w_i) \in \Delta \times \Delta \cong V_i$ such that, locally near the node,

$$(85) \quad D_{i-1,i} = \{w_i = 0\}$$

$$(86) \quad D_{i,i+1} = \{z_i = 0\}$$

We may define certain 1-cycles:

$$(87) \quad \Gamma_{z_i} = \{(z_i, w_i) = (e^{-i\theta}, 1) \mid \theta \in [0, 2\pi)\}$$

$$(88) \quad \Gamma_{w_i} = \{(z_i, w_i) = (1, e^{-i\theta}) \mid \theta \in [0, 2\pi)\}$$

The local model for $U = Y \setminus D$ near the node is simply $(z_i, w_i) \in \Delta^* \times \Delta^*$, which is fibered by the tori $T_{r,s} = \{|z_i| = r, |w_i| = s\}$ where $0 < r, s \leq 1$. The loops Γ_{z_i} and Γ_{w_i} are a basis for the first homology of this neighborhood in U .

Remark 2. Note that the orientation is such that Γ_{z_i} winds *clockwise* in the z_i -plane around $\{z_i = 0\} = D_{i,i+1}$. In a later section this will be justified by the observation that Reeb orbits near the node are homologous to non-negative linear combinations of Γ_{z_i} and Γ_{w_i} .

5.3. The local model for divisors. The divisor D_i has a tubular neighborhood isomorphic to a disk bundle DN_i in the normal bundle $N_i \cong \mathcal{O}(k_i)$. Let SN_i denote the boundary of this tubular neighborhood; it is a circle bundle over D_i and is diffeomorphic to a lens space $L(-k_i, 1)$. Two points on $D_i = D_{i,i+1}$ are distinguished: the i th and $(i+1)$ th nodes. Let $W_{i,i+1}$ denote the complement of V_i and V_{i+1} (the neighborhoods of the nodes) in DN_i :

$$(89) \quad W_{i,i+1} = DN_i \setminus (V_i \cup V_{i+1}).$$

Thus $W_{i,i+1}$ is a disk bundle over an annulus, namely the 2-sphere $D_{i,i+1}$ with two disks removed at the nodes, which we denote by $A_{i,i+1}$. The associated circle bundle $SW_{i,i+1} = W_{i,i+1} \cap SN_i$ is the lens space SN_i with two solid tori removed, and it is non-canonically diffeomorphic to a product $T^2 \times I$. The complement of D in this neighborhood $W_{i,i+1} \setminus D_{i,i+1}$ deformation retracts onto the circle bundle, so it has the same first homology.

The circle fibration gives a long exact sequence in homotopy groups, which reduces to

$$(90) \quad 0 \rightarrow \pi_1(F) \rightarrow \pi_1(SW_{i,i+1}) \rightarrow \pi_1(A_{i,i+1}) \rightarrow 0$$

where F is the circle fiber over the some basepoint in $A_{i,i+1}$. As all three fundamental groups are abelian, this sequence holds with π_1 replaced by H_1 , and we will use this notation from now on to sidestep concerns over basepoints. The sequence (90) is split, but not canonically. In fact, we have two geometric splittings, induced by the bases $(\Gamma_{z_i}, \Gamma_{w_i})$ and $(\Gamma_{z_{i+1}}, \Gamma_{w_{i+1}})$ at the two nodes, as we shall now elaborate.

Near the i th node, $D_{i,i+1}$ is given by the equation $z_i = 0$. Thus Γ_{z_i} is a loop which links the divisor $D_{i,i+1}$, and hence is homologous to the circle fiber of $SW_{i,i+1}$ equipped with some orientation. Similarly, near the $(i+1)$ th node, $D_{i,i+1}$ is given by the equation $w_{i+1} = 0$, and so $\Gamma_{w_{i+1}}$ is a loop which links the divisor, and is also homologous to the circle fiber equipped with some orientation. In fact these orientations agree, as each is a loop in the fiber of the normal bundle which encircles zero clockwise with respect to the natural orientation of the fibers of the normal bundle as complex lines. Thus:

$$(91) \quad \Gamma_{z_i} \sim f \sim \Gamma_{w_{i+1}}$$

where $f \in H_1(F)$ is the class of the fiber with appropriate orientation.

Near the i th node the loop Γ_{w_i} projects onto $D_{i,i+1}$ as a loop encircling the i th node clockwise with respect to the complex orientation on $D_{i,i+1}$. Near the $(i+1)$ th node, the loop $\Gamma_{z_{i+1}}$ projects onto $D_{i,i+1}$ as a loop encircling the $(i+1)$ th node clockwise. So together these loops form the oriented boundary of the annulus $A_{i,i+1}$, and we find

$$(92) \quad -\pi_* \Gamma_{w_i} \sim b \sim \pi_* \Gamma_{z_{i+1}}$$

where $\pi : SW_{i,i+1} \rightarrow A_{i,i+1}$ is the circle bundle over the annulus and $b \in H_1(A_{i,i+1})$ is an appropriate generator.

Thus we find that

$$(93) \quad -\Gamma_{w_i} \sim \Gamma_{z_{i+1}} + \alpha f$$

in $H_1(SW_{i,i+1})$ for some constant α to be determined. In fact, $\alpha = k_i$ as we shall now show.

An elementary computation with the clutching functions of $N_i \cong \mathcal{O}_{\mathbb{P}^1}(k_i)$ shows how α is related to k_i . Write \mathbb{P}^1 as the union of two complex coordinate charts $U = \{\zeta \in \mathbb{C}\}$, $V = \{\eta \in \mathbb{C}\}$, glued by the correspondence $\zeta = 1/\eta$. Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(p)$ be a line bundle of degree p . Then we have local trivializations

$$(94) \quad (\zeta, \xi) \in U \times \mathbb{C} \cong \mathcal{L}|_U$$

$$(95) \quad (\eta, \gamma) \in V \times \mathbb{C} \cong \mathcal{L}|_V$$

$$(96) \quad (\zeta, \xi) = (1/\eta, \zeta^p \gamma)$$

For $p \geq 0$, one verifies that the sections s given by

$$(97) \quad \xi = (s|_U)(\zeta) = \zeta^r$$

$$(98) \quad \gamma = (s|_V)(\eta) = \eta^{p-r}$$

for $0 \leq r \leq p$ are valid holomorphic sections, so this line bundle really does have degree p .

Let $\zeta = \ell(\theta) = e^{i\theta}$ be a loop in \mathbb{P}^1 . We may lift this loop to the U -trivialization as $(\zeta, \xi) = \ell'(\theta) = (e^{i\theta}, 1)$, and to the V -trivialization as $(\eta, \gamma) = \ell'(\theta) = (e^{-i\theta}, e^{-p i \theta})$ (these loops are geometrically the identical). On the other hand, we have the lift $\ell''(\theta)$ given by $(\zeta, \xi) = \ell''(\theta) = (e^{i\theta}, e^{p i \theta})$ and $(\eta, \gamma) = \ell''(\theta) = (e^{-i\theta}, 1)$

Let $f(\tau) = (1, e^{-i\tau})$ be a loop in the fiber over the point $\zeta = \eta = 1$, which encircles the zero-section clockwise. Let $Z = \mathcal{L}|_{U \cap V} \setminus (U \cap V)$ be the complement of the fibers over $\zeta = 0, \infty$ and the zero-section. Then $H_1(Z; \mathbb{Z}) = \mathbb{Z}^2$, and the classes $[\ell'], [\ell'']$ and $[f]$ are elements of this group. Evidently, we have the relation

$$(99) \quad [\ell'] = [\ell''] + p[f]$$

The space Z is homotopy equivalent to $SW_{i,i+1}$, taking $p := k_i$. Under this correspondence, $[f]$ corresponds to the class of f as in equation (91), $[\ell']$ corresponds to the class of

$-\Gamma_{w_i}$, and $[\ell'']$ corresponds to the class of $\Gamma_{z_{i+1}}$. Comparing equations (93) and (99), we see that $\alpha = p = k_i$.

5.4. Affine charts and gluing. For each node, indexed by $i \in \mathbb{Z}/m\mathbb{Z}$, define an integral cone Q_i that is the non-negative span of Γ_{z_i} and Γ_{w_i} :

$$(100) \quad Q_i := \{a[\Gamma_{z_i}] + b[\Gamma_{w_i}] \mid a, b \in \mathbb{Z}_{\geq 0}\}$$

These sums may be interpreted as classes in $H_1(V_i \setminus D; \mathbb{Z})$, where we recall that V_i is the neighborhood of the node.

The analysis of §5.3 indicates how to glue Q_i and Q_{i+1} . Equations (91) and (93) tell us that

$$(101) \quad a\Gamma_{z_i} + b\Gamma_{w_i} \sim a'\Gamma_{z_{i+1}} + b'\Gamma_{w_{i+1}} \iff \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -k_i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

where the homological relation in equation (101) holds in $SW_{i,i+1}$. When we identify the lattices $H_1(V_i \setminus D; \mathbb{Z})$ and $H_1(V_{i+1} \setminus D; \mathbb{Z})$ using this linear transformation, we find that the images of Q_i and Q_{i+1} intersect along the ray spanned by $\Gamma_{z_i} \sim \Gamma_{w_{i+1}}$:

$$(102) \quad Q_i \cap Q_{i+1} = \mathbb{Z}_{\geq 0} \cdot [\Gamma_{z_i}] = \mathbb{Z}_{\geq 0} \cdot [\Gamma_{w_{i+1}}]$$

Thus we may glue Q_i to Q_{i+1} along this common edge to define an integral linear structure on the union $Q_i \cup Q_{i+1}$. This structure is the one induced by embedding the two cones into a common lattice as above.

There is another characterization of this linear structure in terms of the intersection form of Y . A integral linear structure is determined by the corresponding sheaf of integral linear functions. An integral linear function $f : Q_i \cup Q_{i+1} \rightarrow \mathbb{Z}$ is determined by three numbers $\alpha = f(\Gamma_{w_i})$, $\beta = f(\Gamma_{z_i}) = f(\Gamma_{w_{i+1}})$ and $\gamma = f(\Gamma_{z_{i+1}})$, as the linear structure within each cone is standard. In order for f to be linear we need

$$(103) \quad \alpha = f(\Gamma_{w_i}) = f(-\Gamma_{z_{i+1}} - k_i \Gamma_{w_{i+1}}) = -\gamma - k_i \beta$$

Or in other words $\alpha + k_i \beta + \gamma = 0$. Recalling that $k_i = D_i^2$, and D_{i-1} and D_{i+1} are transverse to D_i , this is equivalent to the orthogonality condition

$$(104) \quad (\alpha D_{i-1} + \beta D_i + \gamma D_{i+1}) \cdot D_i = 0$$

We define an singular integral linear manifold U^{trop} to be the union of the cones Q_i , glued along edges as above. This manifold has $Q_i \cup Q_{i+1}$ as charts, but there is no way to extend the linear (or even affine) structure to the origin (the triple overlaps of the charts), so we simply regard that as a singularity. As the manifold U^{trop} is defined over \mathbb{Z} , we use the notation $U^{\text{trop}}(\mathbb{Z})$ or $U^{\text{trop}}(\mathbb{R})$ to denote integral and real points respectively. The real points $U^{\text{trop}}(\mathbb{R}) \setminus \{0\}$ is a real affine manifold in the usual sense. A special feature of the surface case is that $U^{\text{trop}}(\mathbb{R})$ (with the singular point included) is actually a topological manifold homeomorphic to \mathbb{R}^2 .

A locally linear function $f : U^{\text{trop}} \setminus \{0\} \rightarrow \mathbb{Z}$ is determined by its values on the rays $\alpha_j = f(\Gamma_{z_j})$ and the condition of global linearity is equivalent to the orthogonality condition

$$(105) \quad (\forall i) \quad \left(\sum_{j=1}^m \alpha_j D_j \right) \cdot D_i = 0$$

6. THE DIFFERENTIAL ON SYMPLECTIC COHOMOLOGY

In this section we compute the differential on symplectic cohomology and prove theorem 1.1.

6.1. Generators of symplectic cohomology. We continue the analysis of section 4, picking up with the output of Step 5, a Liouville structure on \overline{U}' such that the boundary contact form is toroidally symmetric and the Liouville class is locally convex.

6.1.1. Periodic orbits. Recalling section 4.3.2, we have that the Reeb vector field R is vertical with respect to the torus fibration, and it is locally torus symmetric. Thus the Reeb flow translates each torus by some amount. If $R = R_1\partial_{\theta_1} + R_2\partial_{\theta_2}$ is the Reeb vector field and $\alpha = f(s)d\theta_1 + g(s)d\theta_2$ is the contact form, recall that $f'R_1 + g'R_2 = 0$ determines the direction of the Reeb field. Thus the torus $\pi^{-1}(s)$ is periodic whenever

$$(106) \quad A(s)^\perp = \left\{ \varrho \in H_1(\pi^{-1}(s), \mathbb{R}) \mid \int_\varrho A(s) = 0 \right\} \subset H^1(\pi^{-1}(s), \mathbb{R})$$

is a rational subspace. Each such periodic torus leads to a T^2 of simple (not multiply covered) Reeb orbits, and an $\mathbb{N}^\times \times T^2$ family of periodic Reeb orbits, where \mathbb{N}^\times keeps track of multiplicity.

Recall from section 5.4 that there is an integral cone $Q_i \subset H^1(T^2, \mathbb{R})$ associated to each node i .

Lemma 6.1. *As the parameter s varies in the neighborhood of the node i , the space $A(s)^\perp$ passes through each rational direction in Q_i exactly once. Hence each primitive integral point in Q_i corresponds to a periodic torus, and each integral point corresponds to periodic torus and a particular multiplicity.*

Proof. At some point along each divisor, the line $A(s)^\perp$ is spanned by the class of the normal circle to the divisor. These correspond to the rays in the affine manifold. The fact that $A \wedge A' < 0$ and $A' \wedge A'' < 0$ implies that the subspace A^\perp rotates monotonically, showing that rational directions are never repeated. \square

Recall from section 2.1 that when we complete \overline{U}' along the boundary Σ , we get a manifold $M = \overline{U}' \cup_\Sigma \Sigma \times [0, \infty)$ with a cylindrical end. The Hamiltonian $H = (e^\rho)^2/2$ has periodic orbits that correspond to Reeb orbits, so that a periodic Reeb orbit of period T corresponds to a periodic orbit of X_H of period 1 sitting in the hypersurface $e^\rho = T$. Thus we have

Corollary 6.2. *The time-one-periodic orbits of the Hamiltonian $H = (e^\rho)^2/2$ in the cylindrical end form an a disjoint union of tori, and these tori are in bijective correspondence with the integral points of the affine manifold $U^{\text{trop}}(\mathbb{Z})$.*

We index the tori of periodic orbits by pairs $p = (s, r)$, where s is a point where the Reeb vector is rational, and $r \in \mathbb{N}^\times$ is a positive integer giving the multiplicity of iteration of the orbit. The torus of periodic orbits is called $T_{s,r}$ or T_p . See Figure 5.

6.1.2. Maslov and Conley-Zehnder indices.

Proposition 6.3. *The Lagrangian tori L near the boundary divisor have vanishing Maslov class. Moreover, the construction of section 4 can be done so as to guarantee that the crossing tori are special Lagrangian.*

Proof. Recall that D is an reduced anticanonical divisor, which means that there is a holomorphic volume form Ω on $Y \setminus D$ that has simple poles along D . In a local coordinate chart (z_1, z_2) near a node, Ω and L have the form

$$(107) \quad \Omega = f(z) \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}, \quad L = \{|z_1| = \epsilon_1, |z_2| = \epsilon_2\}$$

where $f(z)$ is holomorphic and non-vanishing along D . For simplicity orient L by $\nu = d\theta_1 \wedge d\theta_2$. Therefore, the phase of L at $(z_1, z_2) = (\epsilon_1 e^{i\theta_1}, \epsilon_2 e^{i\theta_2})$ is that of

$$(108) \quad \Omega|_L = f(z_1, z_2)(i d\theta_1) \wedge (i d\theta_2) = -f(z_1, z_2) \nu$$

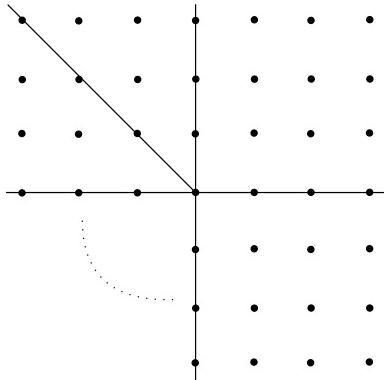


FIGURE 5. Integral points in the affine manifold. The axes indicate quadrants corresponding to nodes.

Since the function f is nonfading for small (z_1, z_2) , the argument function $\arg(f|_L) : L \rightarrow \mathrm{U}(1)$ admits a lift to \mathbb{R} , and so does $\arg(\Omega|_L)$.

We can make these tori special Lagrangian if we assumed that $f(z) \equiv 1$ with respect to the holomorphic coordinate charts near the nodes that were used in Step 1 to make the Kähler form standard.

Since all the tori L near the divisor are Lagrangian isotopic to such a crossing torus, they all also have vanishing Maslov class. \square

Just as in the case of the complex torus $(\mathbb{C}^\times)^2$, after a small non-autonomous perturbation is added to the Hamiltonian, each torus of periodic orbits breaks into several non-degenerate orbits. In fact, locally each torus of periodic torus looks like one in the case of $(\mathbb{C}^\times)^2$, so we can use that case as a local model to understand the perturbation, and in particular the degrees of the generators.

Proposition 6.4. *A small perturbation of the Hamiltonian breaks the torus T_p of periodic orbits in to several non-degenerate orbits. There are at least $\binom{2}{k}$ orbits of Conley–Zehnder index k , for $k = 0, 1, 2$. The Floer trajectories connecting these orbits that stay in a neighborhood of the torus yield a differential whose cohomology can be identified with the ordinary cohomology of the torus $H^*(T_p)$.*

Proof. All of the statements to be proved are local near the Reeb periodic torus L underlying the torus of periodic orbits T_p . In fact, a neighborhood of L in the contact hypersurface Σ can be embedded into a contact hypersurface in $(\mathbb{C}^\times)^n$ (this may require deforming the boundary contact structure of $(\mathbb{C}^\times)^n$, but the description of symplectic cohomology as in section 2.4 still goes through). Thus we see that the local contributions to the Floer differential yield a cohomology of $H^*(T_p)$ just as in the case of $(\mathbb{C}^\times)^2$.

The statement about the Conley–Zehnder indices requires us to further identify the homotopy classes of trivializations of the canonical bundle, locally near the torus L . This holds good because the torus L has Maslov index zero in either trivialization, and L carries all the topology of this neighborhood. \square

This argument about the contribution to the Floer differential that is local to a torus of periodic orbits can be formulated in terms of a spectral sequence associated to the action filtration. The first differential counts holomorphic curves of energy so small that they remain close to the torus of orbits, and so as in the previous proposition contributes $H^*(T_p)$.

The E_1 -page is thus

$$(109) \quad E_1^* = H^*(U) \oplus \left(\bigoplus_{p \in U^{\text{trop}}(\mathbb{Z}) \setminus \{0\}} H^*(T_p) \right)$$

The higher differentials in the spectral sequence count inhomogeneous pseudo-holomorphic curves connecting different critical manifolds. Define $\theta_0 = 1 \in H^0(U)$ and $\theta_p = PD[T_p] \in H^0(T_p)$ for $p \in U^{\text{trop}}(\mathbb{Z}) \setminus \{0\}$. We now reformulate theorem 1.1.

Theorem 6.5. *The degree 0 part of the spectral sequence degenerates at E_1 , so*

$$(110) \quad SH^0(U) \cong E_1^0 = \text{span}\{\theta_p \mid p \in U^{\text{trop}}(\mathbb{Z})\}$$

The preceding results are formulated in terms of a Hamiltonian that is quadratic at infinity. If we use a Hamiltonian H^m that is linear at infinity, analogous results hold for the Floer homology $HF^*(H^m)$. The difference is that only the periodic tori whose Reeb length $\ell_p = \int_{\gamma_p} \alpha$ satisfies $\ell_p < m$ contribute to $HF^*(H^m)$. There is a spectral sequence converging to $HF^*(H^m)$ whose E_1 page is

$$(111) \quad E_1 = H^*(U) \oplus \left(\bigoplus_{p, \ell_p < m} H^*(T_p) \right)$$

Theorem 6.6. *The degree 0 part of the spectral sequence for $HF^*(H^m)$ degenerates at E_1 , so*

$$(112) \quad HF^0(H^m) \cong E_1^0 = \text{span}\{\theta_p \mid p \in U^{\text{trop}}(\mathbb{Z}), \ell_p < m\}$$

For the rest of the computation, we will work with the linear Hamiltonians. This is because the computation of $HF^0(H^m)$ requires us to choose perturbations with favorable properties, and it is not necessarily possible to choose these perturbations in a way that works for the entire cochain complex associated to the quadratic Hamiltonian. Furthermore, as the asymptotic slope m increases, new generators are added to the complex, and we may need to modify the perturbations in order to accommodate these new generators into the method used to compute the differential.

This choice to work with linear Hamiltonians then requires us to analyze the continuation maps $HF^*(H^m) \rightarrow HF^*(H^{m'})$ for $m \leq m'$. We show in the last subsection that these maps are injective on HF^0 , and therefore that, in the colimit used to construct $SH^0(U)$ from $HF^0(H^m)$, all classes at each stage survive to the limit, finishing the proof of 6.5.

6.2. Holomorphic cylinders. The basis for our method of computation is the following nonexistence result. It is proved by adapting a method of Bourgeois and Colin [7]. Throughout this subsection, it may be helpful to have in mind Figure 6 that depicts the projection to Σ of a holomorphic curve in relation to the torus fibration $\pi : \Sigma \rightarrow S$.

Theorem 6.7. *Let (Σ, α) be as above, so that the Liouville class is locally convex. Let $f : C \rightarrow \Sigma$ be a pseudo-holomorphic curve in Σ with finite energy, where the domain Riemann surface C has genus zero and one positive puncture and any number of negative punctures. Then f is trivial, that is, the image of f is a closed integral curve of the Reeb field R .*

Proof. By definition, the map f satisfies

$$(113) \quad J_\xi \circ \pi_\xi \circ df = \pi_\xi \circ df \circ j$$

where j is the complex structure on the domain C , $\pi_\xi : T\Sigma \rightarrow \xi$ is the projection with kernel spanned by the Reeb vector, and J_ξ is the almost complex structure on the contact

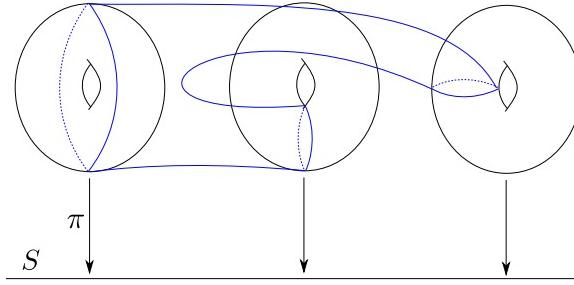


FIGURE 6. The projection to Σ of a holomorphic curve in relation to the projection $\pi : \Sigma \rightarrow S$.

distribution. Consider the two form $f^*d\alpha$ on C . By proposition 2.5, $f^*d\alpha$ is non-negative and only vanishes at points where df maps TC into the line spanned by R .

We furthermore consider the composition of f with the projection to the base $\pi : \Sigma \rightarrow S$. We first show that if $\pi \circ f$ is constant then the conclusion of the theorem follows. Observe that the tangent map $d\pi$ maps ξ surjectively onto TS with a one-dimensional kernel. If $\pi \circ f$ is constant, then $df(v)$ lies in $\ker d\pi$ for every v , and so $\pi_\xi \circ df(v)$ lies in the line bundle $\ker d\pi \cap \xi$ for every v . Thus $\pi_\xi \circ df(v)$ and $\pi_\xi \circ df(jv)$ are proportional, and their symplectic pairing under $d\alpha$ vanishes. Thus $\pi_\xi \circ df(v) = 0$ for every v . Hence $df(v)$ is always proportional to R , and the image of f must necessarily be a closed integral curve of R , which was to be shown.

It remains to rule out the possibility that $\pi \circ f$ is non-constant. This is due to a *local* energy obstruction to the existence of maps with certain topologies. We have split off this part of the argument into the following lemma, which completes the proof of the theorem. \square

Lemma 6.8. *Let f satisfy the hypothesis of theorem 6.7. Then the composition $\pi \circ f$ is constant.*

Proof. Suppose that the projection $\pi \circ f$ is non-constant. Then since map f has finite energy, it is asymptotic at the punctures to closed Reeb orbits. Each Reeb orbit lies entirely within a torus fiber of π . Observe that $\pi \circ f$ is asymptotic at the puncture to the point in S over which this Reeb orbit lies, and thus $\pi \circ f$ may be extended to a map whose domain is \mathbb{CP}^1 . Hence a small loop around the puncture maps to a null-homotopic loop in S . As the fundamental group of C is generated by such loops, we see that $\pi \circ f$ is null-homotopic. If $\tilde{\pi} : \tilde{\Sigma} \rightarrow \tilde{S}$ is the pull back of the torus fibration to the universal cover of the base, we find that f lifts to $\tilde{f} : C \rightarrow \tilde{\Sigma}$. Now consider the projection of $\tilde{\pi} \circ \tilde{f} : C \rightarrow \tilde{S}$, and note $\tilde{S} \cong \mathbb{R}$. By our observation about the punctures, we find that $\tilde{\pi} \circ \tilde{f}$ has a global maximum and minimum. Since $\pi \circ f$ is assumed non-constant, and there is only one positive puncture, at most one of these global extrema can be the limit of a positive puncture. So we obtain a point $s_0 \in \tilde{S}$ which is a global extreme value of $\tilde{\pi} \circ \tilde{f}$, and which is not the limit of a positive puncture.

Now we claim that s_0 must be the limit at a negative puncture. If not, then let $s_1 \in \tilde{S}$ be a regular value of $\tilde{\pi} \circ \tilde{f}$ that lies between s_0 and the nearest point which is a limit of a puncture. Then $P = (\tilde{\pi} \circ \tilde{f})^{-1}[s_0, s_1] \subset C$ is a compact submanifold whose boundary is a collection of smooth curves. These boundary curves lie in the fiber $\tilde{\pi}^{-1}(s_1)$. We claim

$$(114) \quad \int_P d\alpha = \int_{\partial P} \alpha = 0$$

This follows once we show that ∂P is null-homologous in the fiber torus $\tilde{\pi}^{-1}(s_1)$, since α is closed on this torus. But indeed, the capping surface P can be pushed entirely inside of this

fiber, since the torus fibration over the interval $[s_0, s_1]$ is trivial. Now we have that P is a holomorphic curve with vanishing $d\alpha$ -area. Thus P must be contained in a closed integral curve of R , contradicting the construction of P .

The only remaining possibility is that the extremum s_0 is the limit at a negative puncture. In the contact manifold $\tilde{\Sigma}$, the map \tilde{f} is asymptotic to a closed Reeb orbit. Since the puncture is negative, the orientation of this curve by the vector field R is *opposed* to the orientation of this curve as the boundary of \bar{C} (where we compactify C to \bar{C} by adding a circle around every puncture). Again choose a regular value s_1 between s_0 and the nearest point which is a limit of a puncture. Then $P = (\tilde{\pi} \circ \tilde{f})^{-1}[s_0, s_1] \subset \bar{C}$ is a compact surface in \bar{C} whose boundary consists of some circles $\partial_1 P \subset \tilde{\pi}^{-1}(s_1)$, along with some circles in the fiber $\partial_0 P \subset \tilde{\pi}^{-1}(s_0)$. We orient the boundary so that $\partial P = \partial_1 P - \partial_0 P$, and then the circles comprising $\partial_0 P$ are all geometrically Reeb orbits with the Reeb orientation.

Because, as before, the entire surface P can be pushed into one fiber of $\tilde{\pi}$, we find that each of the regular level sets $(\tilde{\pi} \circ \tilde{f})^{-1}(s)$ represents the same homology class in the torus fiber for $s_0 < s < s_1$. Call this class ϱ . Looking at $\partial_0 P$, we see that this class ϱ is a strictly positive multiple of the class of the primitive Reeb orbit in the fiber $\tilde{\pi}^{-1}(s_0)$.

Now we shall apply lemma 4.3. Consider the “period integral”

$$(115) \quad I(s) = \int_{\varrho} A(s)$$

By applying lemma 4.3 to ϱ , we find that $I(s)$ has a non-degenerate local maximum at $s = s_0$. Now the $d\alpha$ -area of f is

$$(116) \quad \int_P d\alpha = \int_{\partial_1 P} \alpha - \int_{\partial_0 P} \alpha = \int_{\varrho} A(s_1) - \int_{\varrho} A(s_0) = I(s_1) - I(s_0) < 0$$

which is impossible. \square

Corollary 6.9. *Let $u : C \rightarrow \Sigma \times \mathbb{R}$ be a pseudo holomorphic curve in the symplectization of Σ (where the complex structure on $\Sigma \times \mathbb{R}$ is cylindrical. Then u is trivial, that is, the image of its projection to Σ consists of a closed integral curve of R .*

Let $H : \Sigma \times \mathbb{R} \rightarrow \mathbb{R}$ be a function that depends only on the \mathbb{R} component. Let $u : C \rightarrow \Sigma \times \mathbb{R}$ be an inhomogeneous pseudo holomorphic curve with Hamiltonian H . Then u is trivial.

Proof. Clear from propositions 2.3 and 2.4. \square

6.3. Neck stretching. It is possible to combine these nonexistence results with an SFT-style neck-stretching argument to prove nonexistence in other situations. This neck-stretching technique was introduced by Bourgeois and Oancea [8, §5] in order to construct an exact sequence relating symplectic cohomology and contact homology. We comment that the main result of Bourgeois and Oancea has stringent technical hypotheses (expected to be alleviated using the polyfold theory of Hofer-Wysocki-Zehnder) related to transversality of holomorphic curve moduli spaces. However, in the present paper, we only use the “compactness” direction of their argument, and we neither “count” nor “glue” holomorphic curves in any situation where transversality cannot be achieved by perturbation of H and J .

Remark 3. Diogo’s thesis [11], as well as forthcoming works of Borman and Diogo-Lisi expand on the idea of using the neck-stretching process to understand symplectic cohomology, in situations relevant to the present paper, namely, manifolds obtained as the complement of symplectic hypersurface in a compact symplectic manifold.

For symplectic cohomology, we have a Hamiltonian $H : U \rightarrow \mathbb{R}$, which is zero in the interior and depends only on ρ on the end, and its time-dependent perturbation $K : S^1 \times U \rightarrow \mathbb{R}$, and we have a time-dependent perturbation of the complex structure $J(t, x)$. We will

consider deformations of this structure in two directions. On the one hand, we will take the perturbations to zero, making the Hamiltonian and complex structure closer to the original unperturbed structures. On the other hand, we will “stretch the neck” along the contact hypersurface. This means that we will take the region where the Hamiltonian is zero to be larger and larger. This forces the periodic orbits in the end to move farther out.

Proposition 6.10. *Let $\mathcal{M}(\gamma_-, \gamma_+)$ be a moduli space of solutions to Floer’s equation on a cylinder, asymptotic to periodic orbits γ_- and γ_+ in the cylindrical end of U , that is counted either by the differential or the BV operator. Suppose that γ_- and γ_+ do not correspond to iterates of the same periodic Reeb torus. Then, for perturbations $K(t, x)$ and $J(t, x)$ sufficiently close to H and J , and for a sufficiently large neck parameter, the moduli space $\mathcal{M}(\gamma_-, \gamma_+)$ is empty.*

Proof. Suppose not, so that there is a sequence of perturbations converging to (H, J) and neck parameters converging to infinity such that $\mathcal{M}(\gamma_-, \gamma_+)$ is always nonempty. Then, after possibly passing to a subsequence, there is a sequence of Floer solutions for each of these parameter values that converges to some a building. The bottom level of this building is a holomorphic curve in U for the complex structure J , the top level is a Floer solution for the data (H, J) in the symplectization $\Sigma \times \mathbb{R}$, and any intermediate levels are holomorphic curves in $\Sigma \times \mathbb{R}$. These curves in these various levels are asymptotic to Reeb orbits in Σ .

Since the original asymptotics γ_- and γ_+ are in the end, there must be at least one symplectization level, and since these orbits do not correspond to iterates of the same periodic Reeb torus, there must be a nontrivial symplectization level. Since the original curves all have the topology of a cylinder, each component of the curve in each level has at most one positive puncture. But the existence of such curves contradicts corollary 6.9 \square

6.4. Computation of differentials. Let θ be a degree zero generator corresponding to the fundamental class of a periodic torus of Reeb orbits. We call θ *primitive* if the corresponding Reeb orbit is primitive (or simple), that is, not an iterate of a closed Reeb orbit. A boundary generator is one coming from the periodic Reeb orbits in the contact boundary, while an interior generator is one coming from the $H^*(U)$ component of the E_1 page.

The strategy for proving $d\theta = 0$ has several ingredients. We use the way that the differential d interacts with the BV operator Δ and the product, namely, that Δ is a chain map and d is a derivation of the product. We also combine this with the nonexistence results for Floer cylinders in section 6.3. All of these results in this section therefore require that the structures be sufficiently close to the neck-stretching limit. Furthermore, we use some aspects of the computation of symplectic cohomology for the complex torus $(\mathbb{C}^\times)^2$. We described the BV operator and product in section 2.4. The general principle here is that the structures of the operations near a single periodic torus are the same as those in the case of $(\mathbb{C}^\times)^2$, as long as we only consider the contributions of holomorphic curves that stay near that torus (the “low-energy” contributions). Of course there may be other contributions to these operations coming from holomorphic curves that extend outside of this neighborhood, but they will have higher energy.

Proposition 6.11. *Let θ be a primitive generator. Then $d\theta$ is a sum of interior generators.*

Proof. We must show the matrix element of the differential connecting θ to any other boundary generator β is zero. This matrix element is a count of inhomogeneous pseudo-holomorphic cylinders connecting θ at the positive puncture to β at the negative puncture. Because θ is primitive, β cannot be an iterate of the same orbit on the same periodic torus as θ . Therefore, by proposition 6.10, this moduli space of cylinders is eventually empty in the neck-stretching limit. \square

Proposition 6.12. *Let θ be a primitive degree zero generator. Then there is a generator η such that $\Delta(\eta) = \theta + (\text{interior generators})$.*

Proof. Let η be the generator corresponding to the same periodic torus as θ , but corresponds to the class in $H^1(T^2)$ Poincaré dual to a loop transverse to the circle action rotating the parametrization. As we saw in the case of $(\mathbb{C}^\times)^2$, the low-energy contribution to the BV operator $\Delta(\eta)$ gives the class swept out by this loop under the circle action, which is the fundamental class of the torus, that is, θ . Thus $\Delta(\eta) = \theta + (\text{higher energy terms})$. Since the BV operator Δ also counts cylinders with a single positive puncture at the primitive orbit θ , in the neck-stretching limit there are eventually no such cylinders that end on a boundary generator. Thus any other terms in $\Delta(\eta)$ must be interior generators. \square

Proposition 6.13. *Let θ be a primitive degree zero generator. Then $d\theta = 0$.*

Proof. Using the fact that $\Delta \circ d + d \circ \Delta = 0$ (as Δ is an odd chain map), and the η provided by the previous proposition, we obtain

$$(117) \quad -\Delta(d\eta) = d\Delta(\eta) = d\theta + d(\text{interior generators})$$

On the other hand, by proposition 6.11, $d\eta = (\text{interior generators})$, and Δ applied to any interior generator is zero. Thus

$$(118) \quad 0 = d\theta + d(\text{interior generators})$$

Now since η is in degree one, an interior generators appearing in this expression also have degree zero. Since we are looking at the E_1 -page of the spectral sequence, the differentials connecting interior generators to each other have already been taken care of, so $d(\text{interior generators}) = 0$ at the E_1 level. Alternatively, if we work at the chain level, we can choose a perturbation in the interior that has a unique minimum, and then there is only one degree zero generator, which is must be closed as it is the identity of symplectic cohomology. \square

Proposition 6.14. *The chain complex for $SH^*(U)$ is generated under the pair-of-pants product by the interior generators and the primitive boundary generators.*

Proof. If θ^n and θ^m are degree zero generators corresponding to the n th and m th iterates of a given primitive orbit θ , then the pair of pants product $\theta^n * \theta^m = \theta^{m+n} + (\text{higher action terms})$. This is again using the principle that the leading order term is the same as in the case of $(\mathbb{C}^\times)^2$. Now we use an action filtration argument (the action spectrum is discrete). Suppose that θ^n is a generator of highest action which is not in the subspace generated by the interior generators and the primitive boundary generators. Then θ^{n-1} is so generated, and in the equation $\theta^{n-1} * \theta = \theta^n + (\text{higher action terms})$, everything except θ^n is so generated as well. Thus θ^n is generated as well, a contradiction. \square

Proposition 6.15. *Let θ^n be a degree zero generator, not necessarily primitive. Then $d\theta^n = 0$.*

Proof. By the previous proposition, we may express θ^n as a linear combination of pair-of-pants-products of primitive generators and interior generators. Since all of the latter are d -closed, and d is a derivation of the product, the result follows. \square

This proposition completes the proof of theorem 6.5.

Remark 4. There is another class in the image of the BV operator, call it $\zeta \in H^1(T_p)$, this is the image of the Poincaré dual of the point class in $H^2(T_p)$. Geometrically it represents a single orbit of the circle action. The conclusion of proposition 6.13 also holds for the class ζ corresponding to a torus of primitive orbits.

6.5. Continuation maps. The last aspect we need to address to tie the calculation together, which are the continuation maps relating the various Floer cohomology groups used in the definition of symplectic cohomology. Recall from section 2.3, one way to define symplectic cohomology is using Hamiltonian functions that are linear at infinity. A Hamiltonian with slope m is denoted by H^m . The prohibited values for m are the lengths of the periodic Reeb orbits in the contact hypersurface Σ , but otherwise we get a Floer cohomology group $HF^*(H^m)$. There are continuation maps $HF^*(H^m) \rightarrow HF^*(H^{m'})$ when $m \leq m'$. These maps are isomorphisms when $m = m'$, even though two different Hamiltonians of the same asymptotic slope are used to define the source and target spaces. The symplectic cohomology is defined as the limit of the directed system constructed from the continuation maps:

$$(119) \quad SH^*(U) \cong \lim_{m \rightarrow \infty} HF^*(H^m)$$

The significance of the continuation maps is that, in the computations of the various groups $HF^*(H^m)$, we must choose perturbations of the Hamiltonian and other structures, and the continuation maps express in a canonical way the invariance of $HF^*(H^m)$, or more precisely, its dependence solely on the asymptotic slope m . Furthermore, as we increase this asymptotic slope, we may need to modify the perturbations used to define the various generators θ_p , and the continuation map from slope m to slope m' expresses in a canonical way how the generator θ_p defined in $HF^*(H^m)$ is related to the generator with the same name in $HF^*(H^{m'})$.

Proposition 6.16. *Let $m \leq m'$. The continuation map $\phi : HF^0(H^m) \rightarrow HF^0(H^{m'})$ is injective. For $p \in U^{\text{trop}}(\mathbb{Z})$, let θ_p^m and $\theta_p^{m'}$ be the degree zero generators in $HF^0(H^m)$ and $HF^0(H^{m'})$ corresponding to the same torus of periodic orbits T_p . There a choice of Hamiltonians that ensures that $\phi(\theta_p^m) = \theta_p^{m'} + (\text{higher action terms})$*

Proof. First observe that, by an action filtration argument, the claim concerning $\phi(\theta_p^m)$ implies the claim that ϕ is an embedding. Indeed, using the elements θ_p^m and $\theta_p^{m'}$, ordered by decreasing levels of action, as basis of the source and target spaces, we find that the matrix of ϕ is upper triangular (a fact which is true for continuation maps in all situations). The claim of the proposition amounts to saying that the matrix elements along the diagonal are equal to one, so that ϕ is an embedding.

Up to this point we have not been particularly specific about our choice of Hamiltonians, since it is largely immaterial, but here we will make a more specific choice. Assume that both H^m and $H^{m'}$ approximate the same quadratic Hamiltonian. By this we mean that H^m is piecewise polynomial function of e^ρ , with a quadratic piece and a linear piece: it is equal to the quadratic $H^Q = h(e^\rho)$ up to the point where $e^\rho = (h')^{-1}(m)$, and it is linear of slope m afterward. As a result the function is C^1 . The Hamiltonian $H^{m'}$ is similar, but, it stays quadratic over a longer interval up to when $e^\rho = (h')^{-1}(m')$. As a result, Hamiltonian periodic tori for H^m are precisely a subset of those for $H^{m'}$, and the actions of these periodic tori are the same when computed using either Hamiltonian.

When computing $HF^0(H^m)$ and $HF^0(H^{m'})$ are perturbed in potentially different ways. We need to show that the generator $\theta_p^{m'}$ appears in $\phi(\theta_p^m)$ with coefficient 1. The cylinders contributing to this matrix element have small action and therefore they are the same as in the case of $(\mathbb{C}^\times)^2$. Since the generators of the form θ_p^m and $\theta_p^{m'}$ represent the same class in $SH^0((\mathbb{C}^\times)^2) \cong H_2(\mathcal{L}T^2)$, this matrix element of this continuation map in the case of $(\mathbb{C}^\times)^2$ must be one. \square

The same idea is also used to relate the Floer cohomologies $HF^0(H^m)$ of Hamiltonians with linear growth to the Floer cohomology $HF^0(H^Q)$ of a Hamiltonian with quadratic

growth. The flow of the quadratic Hamiltonian H^Q creates at once all of the periodic orbits we need to consider, and after perturbation we get degree-zero generators $\theta_p^Q \in CF^0(H^Q)$ corresponding to the torus of periodic orbits T_p . There are continuation maps $\phi : CF^0(H^m) \rightarrow CF^0(H^Q)$ from the finite slope version to the quadratic version. The same argument as in Proposition 6.16 yields:

Proposition 6.17. *For any m , the continuation map $\phi : CF^0(H^m) \rightarrow CF^0(H^Q)$ is injective. If θ_p^m and θ_p^Q are the generators corresponding to a torus of periodic orbits T_p , then there is a choice of Hamiltonians that ensures $\phi(\theta_p^m) = \theta_p^Q + (\text{higher action terms})$.*

Since the continuation map is a chain map, it maps closed elements to closed elements, showing that θ_p^Q is closed, and gives rise to a class in $HF^0(H^Q) = SH^0(U)$. To deduce Theorem 1.1, we take $\theta_p = \theta_p^Q$.

Remark 5. The fact that the continuation maps are triangular with respect to the action filtration (rather than strictly diagonal) is unsatisfying if we want to claim to have found a “canonical basis” for the symplectic cohomology. It seems to the author quite likely that the higher action terms in $\phi(\theta_p^m)$ vanish if the Hamiltonian is chosen correctly. Knowing this would make the result somewhat sharper, in that we would have a more compelling reason to identify the basis elements θ_p with the canonical basis elements of Gross-Hacking-Keel. But even if we were able to prove that, it seems to the author that the only true test of whether these θ_p really are the Gross-Hacking-Keel theta functions will come when one has to match up the product structure on $SH^0(U)$ with the product of theta functions.

7. WRAPPED FLOER COHOMOLOGY

We will now describe a relationship between the symplectic cohomology of U and the wrapped Floer cohomology of certain Lagrangian submanifolds in U , and make a connection with the results of [19].

We consider Lagrangian submanifolds L , which may be either compact or *cylindrical at infinity*, meaning that, within the cylindrical end $\Sigma \times [0, \infty)$ of the completion of our Liouville domain, L has the form $\Lambda \times [0, \infty)$, where Λ is a Legendrian submanifold of Σ .

Given two such Lagrangians L and K , the wrapped Floer cohomology $HW^*(L, K)$ is \mathbb{K} -vector space, which in the situation we consider will be \mathbb{Z} -graded. The definition is parallel to that of symplectic cohomology: We fix a Hamiltonian H . There is a cochain complex $CF^*(L, K; H)$ generated by time-one chords of H starting on L and ending on K (rather than periodic orbits of H). The differential now counts pseudo-holomorphic strips joining such chords, rather than cylinders. If we choose a quadratic Hamiltonian H^Q , we denote this complex by $CW^*(L, K)$, and its cohomology is $HW^*(L, K)$. If we use a linear Hamiltonian H^m of slope m , then as before we need to take a direct limit as $m \rightarrow \infty$.

Using the wrapped Floer cohomology $HW^*(L, K)$ as the space of morphisms from L to K , we obtain the cohomology-level version of the wrapped Fukaya category $\mathcal{W}(U)$. At the chain level, $\mathcal{W}(U)$ is an A_∞ -category [5], which forms the A-side of the homological mirror symmetry correspondence for open symplectic manifolds. In particular, the endomorphisms of a single object, $HW^*(L, L)$, forms a ring. This will be our main object of interest.

To relate symplectic cohomology and wrapped Floer cohomology, we use *closed-to-open string maps* [3]. There are various versions, all defined by counting pseudo-holomorphic curves with boundary, and with a mixture interior punctures (corresponding to generators of symplectic cohomology) and boundary punctures (corresponding to generators of wrapped Floer cohomology). The first of these is a map $\mathcal{CO}_0 : SH^*(U) \rightarrow HW^*(L, L)$. This map fits into a larger structure, a map

$$(120) \quad \mathcal{CO} : SH^*(U) \rightarrow HH^*(CW^*(L, L))$$

where $HH^*(CW^*(L, L))$ denotes the Hochschild cohomology of the A_∞ -algebra $CW^*(L, L)$ (with coefficients in itself). The map \mathcal{CO} is a map of rings [14, Proposition 5.3].

7.1. Lagrangian sections. In the case of an affine log Calabi–Yau surface U with compactification (Y, D) , there is a natural class of Lagrangian submanifolds to consider for wrapped Floer cohomology, namely Lagrangians which are sections of the torus fibration near the divisor. By Lemma 4.1 there is a Legendrian section of the torus fibration $\pi : \Sigma \rightarrow S$ on the contact hypersurface. What we desire is a Lagrangian L that caps off this circle in Σ to a disk in U . For our present purposes, we say that a Lagrangian L is a *section* if

- (1) L is diffeomorphic to a disk, and
- (2) At infinity, L is a cylinder over a Legendrian section of the torus fibration on Σ .

The wrapped Floer cohomology $HW^*(L, L)$ is simple to compute using a quadratic Hamiltonian. Whereas in computing symplectic cohomology we encountered T^2 -families of periodic orbits for the Hamiltonian, chords of the Hamiltonian flow joining L to itself have less symmetry. Since L intersects each torus in Σ in one point, each torus of periodic orbits (in the free loop space) contains exactly one orbit that is a chord from L to L . As these tori are indexed by points $p \in U^{\text{trop}}(\mathbb{Z}) \setminus \{0\}$, denote by $\theta_{L,p}$ the corresponding chord. The chords $\theta_{L,p}$ are generators for $CW^*(L, L)$, and they live in degree zero. There is one more generator $\theta_{L,0} \in CW^0(L, L)$ corresponding to the ordinary cohomology $H^0(L)$, which has rank one since L is a topologically a disk. Because $CW^*(L, L)$ is concentrated in degree zero, the differential vanishes trivially.

Proposition 7.1. *Let L be a Lagrangian section. Then the wrapped Floer complex $CW^*(L, L)$ is concentrated in degree zero, is isomorphic to its cohomology, and has a basis of chords indexed by the points of $U^{\text{trop}}(\mathbb{Z})$.*

$$(121) \quad CW^0(L, L) \cong HW^0(L, L) \cong \text{span}\{\theta_{L,p} \mid p \in U^{\text{trop}}(\mathbb{Z})\}$$

Next we consider the closed-to-open string map $\mathcal{CO}_0 : SH^0(U) \rightarrow HW^0(L, L)$. It is useful to consider once again the case of the complex torus $(\mathbb{C}^\times)^n$ as a model. For the Lagrangian section we take the real positive locus $L = (\mathbb{R}_{>0})^n \subset (\mathbb{C}^\times)^n$. The wrapped Floer cohomology is isomorphic (as a ring) to the space of Laurent polynomials

$$(122) \quad HW^0(L, L) \cong \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

and hence abstractly isomorphic to $SH^0((\mathbb{C}^\times)^n)$. The map \mathcal{CO}_0 implements this isomorphism concretely, by counting disks satisfying an inhomogeneous pseudo-holomorphic map equation with one interior puncture corresponding to the input, one boundary puncture corresponding to the output, and a Lagrangian boundary condition on L . The map \mathcal{CO}_0 sends the degree zero generator in $SH^0((\mathbb{C}^\times)^n)$ corresponding to a torus of periodic orbits to the unique chord in $HW^0(L, L)$ corresponding to the same torus.

In the general situation of a Lagrangian section L in a log Calabi–Yau surface U , the leading order term of the map \mathcal{CO}_0 looks the same as in the case of $(\mathbb{C}^\times)^2$. Thus, given a degree zero generator $\theta_p \in SH^0(U)$, there is a low-energy pseudo-holomorphic curve connecting θ_p to the corresponding generator $\theta_{L,p} \in HW^0(L, L)$. Thus

$$(123) \quad \mathcal{CO}_0(\theta_p) = \theta_{L,p} + (\text{higher action terms})$$

Now a straightforward action filtration argument implies:

Proposition 7.2. *For a Lagrangian section L in an affine log Calabi–Yau surface with maximal boundary U , the map*

$$(124) \quad \mathcal{CO}_0 : SH^0(U) \rightarrow HW^0(L, L)$$

is an isomorphism.

In many cases, it is simple to construct a Lagrangian submanifold with the desired properties. In the toric case, we take $L = (\mathbb{R}_{>0})^2 \subset (\mathbb{C}^\times)^2$. Variations on this work in other cases. When Y is obtained by the blow up of \mathbb{CP}^2 in several points, we can often arrange for the blow up points and the anticanonical divisor to be compatible with the real structure on \mathbb{CP}^2 . Removing the anticanonical divisor will then disconnect the real locus of Y into several components. If done right one of them will be a disk.

For example, in the case of the cubic surface with a triangle of lines (§3.2.2), we can take the blow up points p_1, \dots, p_6 to be real. The anticanonical divisor has the form $D = L_{ab} + E_b + C_a$. If we choose this so that C_a is represented in the real picture as an ellipse, and the region bounded by C_a and L_{ab} contains p_b as the only blow up point on its boundary, then this region is a connected component of the real locus of $Y \setminus D$, which we may take as our Lagrangian section.

In the case of the degree 5 del Pezzo surface (§3.2.3), we blow up \mathbb{CP}^2 in four points p_1, \dots, p_4 , which we take to be real. When we remove an anticanonical 5-cycle, one component of the real locus is a disk. In fact, removing all 10 of the (-1) -curves at once disconnects the real locus (which is a non-orientable surface of Euler characteristic -3) into 12 disks (which are combinatorially pentagons).

Another way to find appropriate Lagrangians is to explicitly consider the Lagrangian torus fibration on the whole of U , rather than just near the boundary, and try to construct a section thereof. Such a torus fibration exists on U due to the existence of a toric model [16], and the results of Symington on almost-toric structures on blow-ups [24]. For example, in the case of a punctured A_n Milnor fiber U (§3.2.4), there is a Lefschetz fibration on $U \rightarrow \mathbb{C}^\times$ whose fibers are affine conics with $n+1$ singular fibers at the points $\xi_k = \exp(2\pi i k / (n+1))$. Following the sort of construction found in [6], we can construct a Lagrangian torus by taking a circle of radius r in the base \mathbb{C}^\times , and looking at the family of circles at some fixed “height” in the conic fibers. Such tori foliate U , and there is one singular fiber, which is a torus with $n+1$ nodes. To construct a Lagrangian section of this torus fibration, begin with a path ℓ in the base \mathbb{C}^\times joining 0 to ∞ , and not passing through any critical value. Over a particular point in ℓ , the fiber is an affine conic, also isomorphic to \mathbb{C}^\times , and we may take again an infinite path joining the two ends. Under the symplectic parallel transport along ℓ , this path in the fiber sweeps out a Lagrangian in the total space, which is our L . If the path ℓ crosses each circle centered at the origin once, and the path in the fiber crosses each circle of constant “height” once, then L is actually a section of the torus fibration.

7.2. The case of the affine plane minus a conic. In [19], the present author considered the case where U is the complement of a smooth conic in \mathbb{C}^2 , which is to say the complement of a conic and a line in \mathbb{P}^2 . In this case the mirror U^\vee is likewise $\mathbb{A}_{\mathbb{K}}^2$ minus a conic (an accident of low dimensions). The ring of global functions on U^\vee is:

$$(125) \quad \mathcal{O}(U^\vee) \cong \mathbb{K}[x, y][(xy - 1)^{-1}]$$

We considered a Lagrangian torus fibration on the whole of U , and a Lagrangian section L . This Lagrangian sections fits into the discussion above, so its wrapped Floer cohomology is concentrated in degree zero and there is no differential. Using techniques particular to this case (and others like it), we computed the ring structure on the wrapped Floer cohomology, and showed that it is isomorphic to the same ring:

$$(126) \quad CW^0(L, L) \cong HW^0(L, L) \cong \mathbb{K}[x, y][(xy - 1)^{-1}]$$

This isomorphism carries the basis elements in $HW^0(L, L)$ corresponding to chords to the functions of the form

$$(127) \quad \{x^a y^b (xy - 1)^c \mid a \geq 0, b \geq 0, c \in \mathbb{Z}\}$$

With this result at hand, we can extend our discussion of the closed-to-open string map. Since we know that $CW^*(L, L)$ is a commutative ring concentrated in degree zero, we find that $HH^0(CW^*(L, L)) \cong CW^0(L, L) \cong \mathcal{O}(U^\vee)$. Since this ring is moreover smooth over \mathbb{K} , the Hochschild–Kostant–Rosenberg theorem implies that

$$(128) \quad HH^p(CW^*(L, L)) \cong H^0(U^\vee, \wedge^p T_{U^\vee})$$

The degree zero piece of the map \mathcal{CO}

$$(129) \quad \mathcal{CO} : SH^0(U) \rightarrow HH^0(CW^*(L, L)) \cong CW^0(L, L)$$

is just the map \mathcal{CO}_0 considered above. Since the map to Hochschild cohomology is naturally a ring map, we can combine this with the previous discussion to obtain that $SH^0(U)$ is isomorphic to $CW^0(L, L)$ as a ring, and thus

$$(130) \quad SH^0(U) \cong CW^0(L, L) \cong \mathcal{O}(U^\vee).$$

The results of [19] can also be used to treat the case of the punctured A_n Milnor fibers (in which the affine manifold has “parallel monodromy-invariant directions”). See [9] for a discussion of related cases.

REFERENCES

1. Alberto Abbondandolo and Matthias Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59** (2006), no. 2, 254–316. MR 2190223 (2006m:53137)
2. ———, *Floer homology of cotangent bundles and the loop product*, Geom. Topol. **14** (2010), no. 3, 1569–1722. MR 2679580 (2011k:53126)
3. Mohammed Abouzaid, *A geometric criterion for generating the Fukaya category*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 112, 191–240. MR 2737980 (2012h:53192)
4. Mohammed Abouzaid, Denis Auroux, and Ludmil Katzarkov, *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*, (2012), preprint arXiv:1205.0053.
5. Mohammed Abouzaid and Paul Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol. **14** (2010), no. 2, 627–718. MR 2602848
6. Denis Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. GGT **1** (2007), 51–91. MR 2386535 (2009f:53141)
7. Frédéric Bourgeois and Vincent Colin, *Homologie de contact des variétés toroïdales*, Geom. Topol. **9** (2005), 299–313 (electronic). MR 2116317 (2005k:53166)
8. Frédéric Bourgeois and Alexandru Oancea, *An exact sequence for contact- and symplectic homology*, Invent. Math. **175** (2009), no. 3, 611–680. MR 2471597 (2010e:53149)
9. Kwokwai Chan and Kazushi Ueda, *Dual torus fibrations and homological mirror symmetry for A_n -singularities*, preprint arXiv:1210.0652.
10. Moira Chas and Dennis Sullivan, *String topology*, preprint, arXiv:math/9911159.
11. Luis Diogo, *Filtered Floer and symplectic homology via Gromov–Witten theory*, Ph.D. thesis, Stanford University, 2012.
12. Jonathan David Evans, *Symplectic mapping class groups of some Stein and rational surfaces*, J. Symplectic Geom. **9** (2011), no. 1, 45–82. MR 2787361 (2012k:53174)
13. Tadao Fujita, *On the topology of noncomplete algebraic surfaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), no. 3, 503–566. MR 687591 (84g:14035)
14. Sheel Ganatra, *Symplectic cohomology and duality for the wrapped Fukaya category*, Ph.D. thesis, Massachusetts Institute of Technology, 2012.
15. Hansjörg Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics, vol. 109, Cambridge University Press, Cambridge, 2008. MR 2397738 (2008m:57064)
16. Mark Gross, Paul Hacking, and Sean Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, preprint: arXiv:1106.4977.
17. Mark Gross and Bernd Siebert, *Theta functions and mirror symmetry*, arXiv:1204.1991.
18. Mark McLean, *The growth rate of symplectic homology and affine varieties*, Geom. Funct. Anal. **22** (2012), no. 2, 369–442. MR 2929069
19. James Pascaleff, *Floer cohomology in the mirror of the projective plane and a binodal cubic curve*, preprint arXiv:1109.3255.
20. Alexander F. Ritter, *Topological quantum field theory structure on symplectic cohomology*, To appear in J. of Topology, preprint arXiv:1003.1781.

21. D. A. Salamon and J. Weber, *Floer homology and the heat flow*, Geom. Funct. Anal. **16** (2006), no. 5, 1050–1138. MR 2276534 (2007k:53154)
22. Paul Seidel, *A biased view of symplectic cohomology*, Current developments in mathematics, 2006, Int. Press, Somerville, MA, 2008, pp. 211–253. MR 2459307 (2010k:53153)
23. Paul Seidel and Jake P. Solomon, *Symplectic cohomology and q -intersection numbers*, preprint arXiv:1005.5156.
24. Margaret Symington, *Four dimensions from two in symplectic topology*, Topology and geometry of manifolds (Athens, GA, 2001), Proc. Sympos. Pure Math., vol. 71, Amer. Math. Soc., Providence, RI, 2003, pp. 153–208. MR 2024634 (2005b:53142)
25. Claude Viterbo, *Functors and computations in Floer cohomology II*, Preprint 2003.
26. R. O. Wells, Jr., *Differential analysis on complex manifolds*, second ed., Graduate Texts in Mathematics, vol. 65, Springer-Verlag, New York, 1980. MR 608414 (83f:58001)
27. Mikhail Zaidenberg, *Lectures on exotic algebraic structures on affine spaces*, arXiv:math/9801075.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN
E-mail address: jpascalleff@math.utexas.edu